COMPARING THE LAPLACIAN WITH AVERAGING OPERATOR
UCONN MATH REU 2023

1. Analytic Approach

Lemma 1. *The difference between the probabilistic Laplacian and the averaging operator applied to a continuous function f on equally spaced points on $[-1,1]$ is bounded. As the number of points goes to infinity, the difference goes to 0.*

Proof. Let $x \in [-1,1]$. Let n be a positive integer. Fix n points equally spaced on $[-1,1]$, denoted $\{x_j\}_{j=0}^{n-1}$. Set δ as the difference between these points. Define $\mathcal{B}(x,\varepsilon) = (x-\varepsilon, x+\varepsilon) \cap [-1,1] - \{x\}$, and $B(x,\varepsilon) = \mathcal{B}(x,\varepsilon) \cap \{x_j\}_{j=0}^{n-1}$.

$$\mathcal{L}_{n,\varepsilon} f(x) = \frac{1}{\#B(x,\varepsilon)} \sum_{x_j \in B(x,\varepsilon)} (f(x) - f(x_j)) = f(x) - \frac{1}{\#B(x,\varepsilon)} \sum_{x_j \in B(x,\varepsilon)} f(x_j)$$

Similarly,

$$\mathcal{L}_{\varepsilon} f(x) = \frac{1}{|\mathcal{B}(x,\varepsilon)|} \int_{\mathcal{B}(x,\varepsilon)} (f(x) - f(y)) d\mu(y) = f(x) - \frac{1}{|\mathcal{B}(x,\varepsilon)|} \int_{\mathcal{B}(x,\varepsilon)} f(y) d\mu(y)$$

Taking the difference,

$$|\mathcal{L}_{n,\varepsilon} f(x) - \mathcal{L}_{\varepsilon} f(x)| = \left| \frac{1}{|\mathcal{B}(x,\varepsilon)|} \int_{\mathcal{B}(x,\varepsilon)} f(y) d\mu(y) - \frac{1}{\#B(x,\varepsilon)} \sum_{x_j \in B(x,\varepsilon)} f(x_j) \right|$$

Now, we want to divide $\mathcal{B}(x,\varepsilon)$ into $\#B(x,\varepsilon)$ pieces $\{I_\ell\}_{\ell=0}^{\#B(x,\varepsilon)-1}$ of equal length $\frac{|\mathcal{B}(x,\varepsilon)|}{\#B(x,\varepsilon)}$. Using this notation,

$$\begin{aligned}
|\mathcal{L}_{n,\varepsilon} f(x) - \mathcal{L}_{\varepsilon} f(x)| &= \left| \frac{1}{|\mathcal{B}(x,\varepsilon)|} \int_{\mathcal{B}(x,\varepsilon)} f(y) d\mu(y) - \frac{1}{\#B(x,\varepsilon)} \sum_{x_j \in B(x,\varepsilon)} f(x_j) \right| \\
&= \frac{1}{\#B(x,\varepsilon)} \left| \sum_{x_j \in B(x,\varepsilon)} (f(x_j)) - \frac{\#B(x,\varepsilon)}{|\mathcal{B}(x,\varepsilon)|} \int_{\mathcal{B}(x,\varepsilon)} f(y) d\mu(y) \right| \\
&= \frac{1}{\#B(x,\varepsilon)} \left| \sum_{x_j \in B(x,\varepsilon)} (f(x_j)) - \frac{\#B(x,\varepsilon)}{|\mathcal{B}(x,\varepsilon)|} \sum_{\ell=0}^{\#B(x,\varepsilon)-1} \int_{I_\ell} f(y) d\mu(y) \right| \\
&= \frac{1}{\#B(x,\varepsilon)} \left| \sum_{x_j \in B(x,\varepsilon)} (f(x_j)) - \sum_{\ell=0}^{\#B(x,\varepsilon)-1} \frac{\#B(x,\varepsilon)}{|\mathcal{B}(x,\varepsilon)|} \int_{I_\ell} f(y) d\mu(y) \right|
\end{aligned}$$

By the MVT, there is some point c_ℓ in each I_ℓ with $c_\ell = \frac{\#B(x,\varepsilon)}{|\mathcal{B}(x,\varepsilon)|}\int_{I_\ell} f(y)d\mu(y)$. Let J be the index of the first $x_j \in B(x,\varepsilon)$, that is, $\min_{x_j\in B(x,\varepsilon)} j = J$. Observe that for $z \in I_\ell, |x_{J+\ell} - z| \leq 3\delta$. Thus

$$\begin{aligned}
|\mathcal{L}_{n,\varepsilon}f(x) - \mathcal{L}_\varepsilon f(x)| &= \frac{1}{\#B(x,\varepsilon)}\left|\sum_{x_j\in B(x,\varepsilon)}(f(x_j)) - \sum_{\ell=0}^{\#B(x,\varepsilon)-1}\frac{\#B(x,\varepsilon)}{|\mathcal{B}(x,\varepsilon)|}\int_{I_\ell} f(y)d\mu(y)\right| \\
&= \frac{1}{\#B(x,\varepsilon)}\left|\sum_{x_j\in B(x,\varepsilon)}(f(x_j)) - \sum_{\ell=0}^{\#B(x,\varepsilon)-1} f(c_\ell)\right| \\
&\leq \frac{1}{\#B(x,\varepsilon)}\sum_{x_j\in B(x,\varepsilon)}\sup_{|x_j-z|<3\delta}|f(x_j)-f(z)| \leq \sup_{x_j\in B(x,\varepsilon),|x_j-z|<3\delta}|f(x_j)-f(z)|
\end{aligned}$$

Since all points chosen are evenly spaced, as n goes to infinity, δ goes to 0. Thus

$$|\mathcal{L}_{n,\varepsilon}f(x) - \mathcal{L}_\varepsilon f(x)| \leq \sup_{x_j\in B(x,\varepsilon),|x_j-z|<3\delta}|f(x_j)-f(z)| \to 0$$

□

2. Revisiting the Taylor Series

Refer to Theorem 1.1. Because we are working in an arbitrary metric space X, there is no guarantee of the existence of addition or multiplication in X, and thus we cannot form the derivatives necessary for a Taylor series in X.

3. Convergence in [-1,1]

Fix $\epsilon \in (0,1)$. For f on $[-1,1]$, define a function $\tilde{f}$ on $[-1-\epsilon, 1+\epsilon]$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-1,1] \\ f(2-x) & x \in (1, 1+\epsilon] \\ f(-2-x) & x \in [-1-\epsilon, -1) \end{cases} \tag{3.1}$$

Lemma 2. *If $f \in C^2$, i.e. if both f' and f'' both exist and are both continuous, and $f'(1) = f'(-1) = 0$, then $\tilde{f} \in C^2$.*

Proof. It is sufficient to check that $\tilde{f}, \tilde{f'}, \tilde{f''}$ are continuous at ± 1. Begin by having

$$\tilde{f}(x) = f(2-x) \iff \tilde{f}'(x) = f'(2-x)\cdot(-1) \iff \tilde{f}''(x) = f''(2-x)\cdot(1)$$

Applying a limit,

$$\lim_{y\to 1^+}\tilde{f}(y) = \lim_{y\to 1^+} f(2-y) \text{ Let } z = 2-y \text{ so } y \to 1^+ \implies z \to 1^-.$$

$$\lim_{z\to 1^-} f(z) = f(1) \checkmark$$

$$\lim_{y\to 1^+}\tilde{f}'(y) = \lim_{y\to 1^+} -f'(2-y) = \lim_{z\to 1^-} -f'(z) = 0 = f'(1) \checkmark$$

$$\lim_{y\to 1^+} \tilde{f}''(y) = \lim_{y\to 1^+} f''(2-y) = \lim_{z\to 1^-} f''(z) = f''(1) \checkmark$$

$\therefore \tilde{f} \in C^2$.

□

Now take n uniformly spaced points on $[-1,1]$. Define

$$\mathcal{L}_\varepsilon f(x) = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(x) - f(y)dy,\ x \in [-1,1].$$

Given $n \in \mathbb{N}$, let $x_j = -1 + \frac{2j}{n-1}$ and $k = \lfloor \epsilon n \rfloor$. Define

$$\mathcal{L}_{n,\varepsilon} f(x) = \frac{1}{-}$$

Lemma 3. *Recall from a previous lemma that* $|\mathcal{L}_\varepsilon f(x) - \mathcal{L}_{n,\varepsilon} f(x)| \leq \omega(f; \frac{6}{n-1})$.

Theorem 3.1. *For fixed* $\epsilon > 0$*, if* f *is continuous then* $\mathcal{L}_{n,\varepsilon} f \to \mathcal{L}_\varepsilon f$*. Moreover, if* f *is* α*-Holder, then*

$$||\mathcal{L}_{n,\varepsilon} f(x) - \mathcal{L}_\varepsilon f(x)||_\infty \leq M_\alpha (\frac{6}{n-1})^\alpha.$$

For convenience in the following Lemma, we note that

$$||\mathcal{L}_{n,\varepsilon} f(x) - \mathcal{L}_\varepsilon f(x)||_\infty \leq M_\alpha (\frac{6}{n-1})^\alpha \iff ||\frac{1}{\epsilon^2}\mathcal{L}_{n,\varepsilon} f(x) - \frac{1}{\epsilon^2}\mathcal{L}_\varepsilon f(x)||_\infty \leq \frac{M_\alpha}{\epsilon^2}(\frac{6}{n-1})^\alpha.$$

Lemma 4. *If* $f \in C^2[-1,1]$ *then*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^2}\mathcal{L}_\varepsilon f(x) = -\frac{f''(x)}{6}.$$

Proof. Recall from Lemma 2 that since $f'(1) = f'(-1) = 0$, $\tilde{f}$ is sufficiently smooth so that we may look at the Taylor series. If $x \in [-1,1]$ then take $x < min(|x-1|, |x+1|)$. Consider the Taylor expansion of $\tilde{f}$,

$$\tilde{f}(y) = \tilde{f}(x) + \tilde{f}'(x)(y-x) + \frac{\tilde{f}''(x)}{2}(y-x)^2 + o(|y-x|^2)$$

So,

$$\tilde{f}(y) - \tilde{f}(x) = \tilde{f}'(x)(y-x) + \frac{\tilde{f}''(x)}{2}(y-x)^2 + o(|y-x|^2)$$

$$\tilde{f}(x) - \tilde{f}(y) = -\tilde{f}'(x)(y-x) - \frac{\tilde{f}''(x)}{2}(y-x)^2 + o(|y-x|^2)$$

$$\int_{x-\epsilon}^{x+\epsilon} \tilde{f}(x) - \tilde{f}(y)dy = -\tilde{f}'(x)\int_{x-\epsilon}^{x+\epsilon}(y-x)dy - \frac{\tilde{f}''(x)}{2}\int_{x-\epsilon}^{x+\epsilon}(y-x)^2dy + \int_{x-\epsilon}^{x+\epsilon} o(|y-x|^2)dy$$

Set $t = y - x$.

$$= -\tilde{f}'(x)\int_{-\epsilon}^{\epsilon} tdt - \frac{\tilde{f}''(x)}{2}\int_{-\epsilon}^{\epsilon} t^2dt + \int_{-\epsilon}^{\epsilon} t^2dt \cdot o(\epsilon^2)$$

$$= -\tilde{f}'(x)\cdot 0 - \tilde{f}''(x)\frac{\epsilon^3}{3} + \int_{-\epsilon}^{\epsilon} t^2dt \cdot o(\epsilon^2)$$

Now to recreate our averaging operator, divide both sides by 2ϵ.

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} \tilde{f}(x) - \tilde{f}(y)dy = -\frac{\tilde{f}''(x)\epsilon^2}{6} + \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} t^2dt \cdot o(\epsilon^2)$$

Observe that the following inequality holds;

$$|\mathcal{L}_\varepsilon f(x) + \frac{\tilde{f}''(x)e^2}{6}| \leq \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} t^2 dt \cdot o(\epsilon^2) = \frac{\epsilon^2}{3} \cdot o(\epsilon^2)$$

So,

$$|\frac{1}{\epsilon^2}\mathcal{L}_\varepsilon f(x) + \frac{\tilde{f}''(x)}{6}| \leq \frac{\epsilon^2}{3} \cdot o(1) = 0 \text{ as } \epsilon \to 0$$

$$\therefore \lim_{\epsilon \to 0} \frac{1}{\epsilon^2}\mathcal{L}_\varepsilon f(x) = -\frac{f''(x)}{6}.$$

□