Laplacian Eigenmaps and Chebyshev Polynomials

Jonathan Kerby-White, Yiheng Su

University of Connecticut

August 1, 2023

Outline

Outline

1. Introduction to Graph Laplacian Operators
2. Formula for Eigenvalues and Eigenfunctions
3. Eigencoordinates and Chebyshev Polynomials
4. Analysis of Robin Problem on $[-1, 1]$
   - Continuous Robin Problem with Uniformly Spaced Points
   - Discrete Robin Problem with Uniformly Spaced Points

Introduction

Eugenmaps:

- Eigenmaps are crucial in analysis, geometry, and machine learning, especially for nonlinear dimension reduction.
- The Laplacian eigenmaps introduced by Belkin and Niyogi are widely used for dimension reduction in data analysis.

Focus of the Work:

- Model situations with uniformly spaced points within intervals are considered.
- Eigenmaps are computed both analytically and numerically.

Investigations:

- Exploring relationships between eigenmaps and orthogonal polynomials.
- Studying the method's dependence and stability when varying the choice of Laplacian.

Definitions of Graph Laplacian Operators when $k = 1$

We define the graph $G = \{V, E\}$ on the interval $[-1, 1]$, where V consists of n equally spaced points $\{x_0 = -1, x_1, \dots, x_{n-1} = 1\}$, and $(x_i, x_j) \in E$ if $|x_i - x_j| = \frac{2}{n-1}$.
Let W_G be the adjacency matrix and D_G be the degree matrix of G.
Then, the regular Laplacian matrix is defined by

$$L_{\text{reg}} := D_G - W_G.$$

The probabilistic Laplacian matrix is defined by

$$L_{\text{prob}} := D_G^{-1}(D_G - W_G).$$

$$L_{\text{reg}} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}, L_{\text{prob}} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

Definitions of Graph Laplacian Operators when $k = 1$

We define graph $G' := \{V, E \cup (x_0, x_{n-1})\}$ from n evenly spaced points on a circle. Then, the periodic Laplacian is defined by

$$L_{\text{per}} := D_{G'} - W_{G'}.$$

$$L_{\text{reg}} = \begin{bmatrix} 2 & -1 & 0 & \cdots & -1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

Outline

Formula for Eigenvalues and Eigenfunctions

1. Periodic Laplacian. $\lambda = 2(1 - cos(\pi\alpha\delta))$ and

$$f(x) = e^{i\pi\alpha x}, \alpha \in \mathbb{Z}$$

2. Regular Laplacian. $\lambda = 2(1 - \cos(\pi\alpha\delta))$ and

$$f(x) = \begin{cases} \cos(\pi\alpha x) & \text{with } c \text{ even} \\ \sin(\pi\alpha x) & \text{with } c \text{ odd} \end{cases},$$

where $\alpha = \frac{c}{\delta+2}, c \in \mathbb{Z}$

3. Probabilistic Laplacian. $\lambda = 1 - \cos(\pi\alpha\delta)$ and

$$f(x) = \begin{cases} \cos(\pi\alpha x) & \text{with } c \text{ even} \\ \sin(\pi\alpha x) & \text{with } c \text{ odd} \end{cases},$$

where $\alpha = \frac{c}{2}$ for $c \in \mathbb{Z}$.

Periodic Laplacian k=1

Theorem

Suppose $L^{per}_{n,k=1}$ *is the periodic Laplacian. The eigenvalues are* $\lambda(\alpha) = 2 - 2\cos(\pi\alpha\delta)$ *where* $\alpha \in \{0, 1, \ldots, n-1\}$. *When n is even, the eigenvalues are exactly* $\lambda(0), \lambda(1), \ldots, \lambda(\frac{n}{2})$. *When n is odd, the eigenvalues are exactly* $\lambda(0), \lambda(1), \ldots, \lambda(\frac{n-1}{2})$.

Periodic Laplacian General k

Theorem

Suppose $L_{n,k}^{per}$ is the periodic Laplacian. The eigenvalues of $L_{n,k}^{per}$ are in the form

$$\lambda = 2(k - \sum_{j=1}^{k} \cos(j\pi\alpha\delta)).$$

The corresponding eigenfunctions are in the form

$$f(x) = e^{i\pi\alpha x},$$

where $\alpha \in \mathbb{Z}$.

Periodic Laplacian General k

Lemma

The period of $\lambda(\alpha) = 2(k - \sum_{j=1}^{k} \cos(j\pi\alpha\delta))$ *is* n.

Theorem

Suppose $L_{n,k}^{per}$ *is the periodic Laplacian with* $k < n$. *The eigenvalues are given by* $\lambda(\alpha) = 2(k - \sum_{j=1}^{k} \cos(j\pi\alpha\delta))$, *where* $\alpha \in \{0, 1, \ldots, n-1\}$.

Note: we do not know the multiplicity of eigenvalues in the general case.

Outline

1. Introduction to Graph Laplacian Operators
2. Formula for Eigenvalues and Eigenfunctions
3. Eigencoordinates and Chebyshev Polynomials
4. Analysis of Robin Problem on $[-1, 1]$
   - Continuous Robin Problem with Uniformly Spaced Points
   - Discrete Robin Problem with Uniformly Spaced Points

Eigencoordinates and Chebyshev Polynomials

Suppose L is a Laplacian matrix, and f_1 and f_ℓ are the eigenfunctions corresponding to the first and ℓ^{th} smallest eigenvalues of L. The eigencoordinates are $(f_1(x), f_\ell(x))$.

We want to investigate the error, E, between our eigencoordinates and the Chebyshev polynomial T_ℓ. The error can be obtained from the following equation:

$$E(x) = f_\ell(x) - T_\ell(f_1(x)).$$

$T_\ell(\sin(y))$

We have a lemma for the formula of the $T_\ell(\sin(y))$.

Lemma

$$T_\ell(\sin(y)) = \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}).$$

The following are the first few Chebyshev polynomials of $\sin(y)$

$$T_0(\sin(y)) = 1$$
$$T_1(\sin(y)) = \sin(y)$$
$$T_2(\sin(y)) = -\cos(2y)$$
$$T_3(\sin(y)) = -\sin(3y)$$
$$T_4(\sin(y)) = \cos(4y)$$

Eigencoordinates and Chebyshev Polynomial

Theorem

$E(x) = f_\ell(x) - T_\ell(f_1(x)) = 0$ *for the regular Laplacian matrix.*

Theorem

$E(x) = f_\ell(x) - T_\ell(f_1(x)) = 0$ *for the probabilistic Laplacian matrix.*

Note: these theorems can be proved using the formula of eigenfunctions for L_{reg} and L_{prob}.

Periodic Laplacian and Chebyshev Polynomial

Theorem

$E(x) = f_\ell(x) - T_\ell(f_1(x)) = 0$ *for the periodic Laplacian matrix.*

Proof idea: L^{per} is of the form $e^{i\pi\ell x}$. λ_ℓ has multiplicity 2 and the corresponding eigenvectors are $\{e^{i\pi\ell x}, e^{-i\pi\ell x}\}$ which is a basis for the eigenspace of λ_ℓ. Let $y = \pi x$. So, $f_\ell(x) = \frac{e^{i\ell y}+e^{-i\ell y}}{2} = \cos(\ell y)$ is also an eigenvector for the eigenspace of λ_ℓ. By the definition of Chebyshev polynomials, we have $T_\ell(cos(y)) = cos(\ell y) = f_\ell(x)$.

Outline

Go to Robin Problem

Currently, we possess three distinct types of graph Laplacian operators: L_{reg}, L_{prob}, and L_{per}. Can these operators be consolidated into a single type, or do they belong to different families of Laplacians?

Continuous Robin Problem with Uniformly Spaced Points

Definition

Suppose u is an eigenfunction of the Laplacian $L_{n\to\infty,k=1}$ with corresponding eigenvalue λ. The Robin Boundary Conditions are

$$\begin{cases} u'' = -\lambda u & \text{on } (-1,1) \\ \partial_n u = \rho u & \text{at } x = \pm 1 \end{cases} \tag{4.1}$$

where ∂_n is the outward normal derivative at $x = \pm 1$.

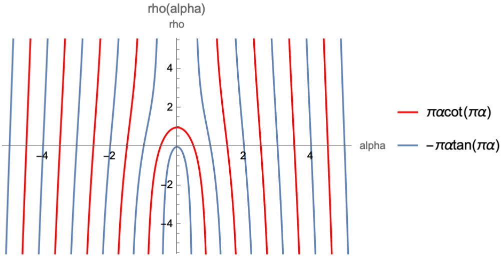
$\rho(\alpha)$

For the equation $e^{2i\pi\alpha} = \pm\frac{i\pi\alpha+\rho}{i\pi\alpha-\rho}$, we have a formula for ρ in terms of α,

$$\rho(\alpha) = i\pi\alpha\frac{(\pm e^{2i\pi\alpha} - 1)}{(\pm e^{2i\pi\alpha} + 1)} = i\pi\alpha\frac{(\pm e^{i\pi\alpha} - e^{-i\pi\alpha})}{(\pm e^{i\pi\alpha} + e^{-i\pi\alpha})}$$

$$= \begin{cases} -\pi\alpha\tan(\pi\alpha), & \text{when } u = \cos(\pi\alpha x) \\ \pi\alpha\cot(\pi\alpha), & \text{when } u = \sin(\pi\alpha x) \end{cases}.$$

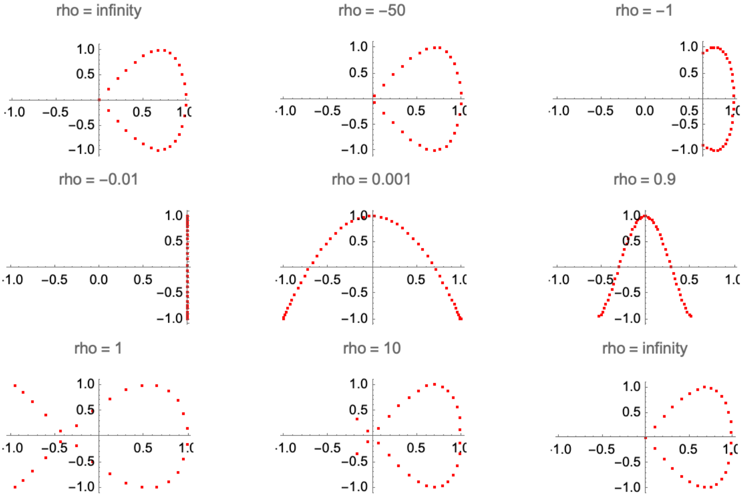
$\rho(\alpha) = 0$



$-\pi\alpha \tan(\pi\alpha) = 0 \iff \alpha = 0$ or $\tan(\pi\alpha) = 0 \iff \alpha \in \mathbb{Z}$.
$\pi\alpha \cot(\pi\alpha) = 0 \iff \alpha = 0$ or $\cot(\pi\alpha) = 0 \iff \alpha \in \mathbb{Z} + \frac{1}{2}$.
Therefore, $\rho(\alpha) = 0 \iff \alpha \in \frac{c}{2}$, where $c \in \mathbb{Z}$. Thus,

$$u = \begin{cases} \sin(\pi \frac{c}{2} x), & \text{when } c \text{ is odd} \\ \cos(\pi \frac{c}{2} x), & \text{when } c \text{ is even} \end{cases}.$$

$\rho(\alpha)$


rho = infinity
rho = −50
rho = −1
rho = −0.01
rho = 0.001
rho = 0.9
rho = 1
rho = 10
rho = infinity

Discrete Robin Problem with Uniformly Spaced Points

Definition

Suppose u is an eigenfunction of the Laplacian matrix $L_{n,k=1}$ of any variety with corresponding eigenvalue λ. The Discrete Robin Boundary Conditions are

$$\begin{cases} Lu = -\lambda u & \text{on } (-1,1) \\ L_{n,k=1}u = \sigma u & \text{at } x = \pm 1 \end{cases}. \tag{4.2}$$

Eigenfunctions for the Discrete Probabilistic Laplacian

Previous work on the continuous Laplacian shows that when $\rho = 0$, the eigencoordinates will be exactly the Chebyshev polynomials.
Hence, $\lim_{n \to \infty} \sigma = 0$ implies that the eigencoordinates for the Probabilistic Laplacian matrix are the Chebyshev polynomials.