GRAPH LAPLACAINS, EIGEN-COORDINATES, CHEBYSHEV POLYNOMIALS, AND ROBIN PROBLEMS
UCONN MATH REU 2023

Contents

1. Glossary of Notation

The following are general notation used in the paper unless stated otherwise:

- The jth point: x_j
- The number of chosen points: n
- The distance between equally spaced points: δ
- The number of points adjacent to x_j on one side: k
- Regular Laplacian: L
- Probabilistic Laplacian: $\mathcal{L}$
- Periodic Laplacian: L^{per}

- $T_\ell(x)$ is the ℓ^{th} Chebyshev polynomial.

  Note: All Laplacians in this document are for $k = 1$ unless stated otherwise.

2. Formula for Eigenvalues and Eigenfunctions

2.1. **Finding Formula for Eigenvalues.** The process of finding eigenvalues of the different Laplacian matrices follows a pattern resulting in the following, helpful lemma.

Lemma 1. *Let L be a Laplacian matrix. If the middle $n-2$ rows are of the form*

$$c\begin{bmatrix} -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & \dots & & -1 & 2 & -1 \end{bmatrix}$$

then for a vector $f(x) = Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}$ to be an eigenvector, its corresponding eigenvalue λ must be $\lambda = 2c(1 - \cos(\pi\alpha\delta))$.

Proof. Let L be a Laplacian matrix of any variety. Suppose that for $1 \le j \le n-2$, the j-th row of L is

$$L(x_j) = c\begin{pmatrix} 0 & \dots & -1 & 2 & -1 & 0 & \dots & 0 \end{pmatrix}$$

where the 2 is at index j. If $f(x) = Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}$ is an eigenvector of L, we have that $L(Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}) = \lambda(Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x})$ for some constant λ. In particular, for $1 \le j \le n-2$, $Lf(x_j) = \lambda f(x_j)$. Note that

$$\begin{aligned} Lf(x_j) &= c(-f(x_{j-1}) + 2f(x_j) - f(x_{j+1})) = c(-f(x_j - \delta) + 2f(x_j) - f(x_j + \delta)) \\ &= c(-(Ae^{i\pi\alpha(x_j-\delta)} + Be^{-i\pi\alpha(x_j-\delta)}) + 2(Ae^{i\pi\alpha x_j} + Be^{-i\pi\alpha x_j}) - (Ae^{i\pi\alpha(x_j+\delta)} + Be^{-i\pi\alpha(x_j+\delta)})) \\ &= c(Ae^{i\pi\alpha x_j}(2 - e^{i\pi\alpha\delta} - e^{-i\pi\alpha\delta}) + Be^{-i\pi\alpha x_j}(2 - e^{i\pi\alpha\delta} - e^{-i\pi\alpha\delta})) \\ &= 2c(1 - \cos(\pi\alpha\delta))(Ae^{i\pi\alpha x_j} + Be^{-i\pi\alpha x_j}) = 2c(1 - \cos(\pi\alpha\delta))f(x_j) \end{aligned}$$

Observe that the coefficient in equation $2c(1-\cos(\pi\alpha\delta))$ is a constant which does not depend on x_j. Thus if $f(x)$ is an eigenvector for L, its corresponding eigenvalue λ must be $\lambda = 2c(1 - \cos(\pi\alpha\delta))$. □

We can verify this numerically. In Figure 2.1 we plot the eigenvalues of the Periodic Laplacian for $n = 40$ and $k = 1$ and we plot the calculated eigenvalues and see that the graphs are qualitatively the same.

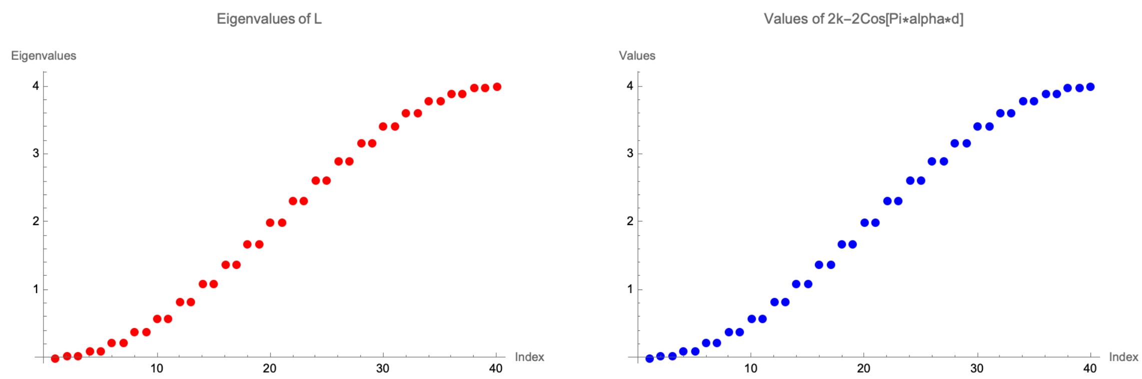

FIGURE 1. The red dots (left) represent the eigenvalues of L^{per} computed by Mathematica where $n = 40$ and $k = 1$. The blue dots (right) represent the eigenvalues of L^{per} computed from the formula $\lambda(\alpha) = 2 - 2\cos(\pi\alpha\delta)$ where $\alpha \in \{0, 1, \dots, 39\}$ with values sorted numerically from smallest to largest.

2.2. **Periodic Laplacian.** For the Periodic Laplacian, we choose n points from a circle for ease of visualization. From there we project those points to the interval $[-1, 1]$ which requires us to choose n points where $x_0 = -1$ and $x_{n-1} = 1$.

Theorem 2.1. *The function $f(x) = e^{i\pi\alpha x}, \alpha \in \mathbb{Z}$ is an eigenfunction for the Periodic Laplacian.*

Proof. Consider the periodic Laplacian, L^{per}, in this case given by,

$$L^{\text{per}}_{j,m} = \begin{cases} 2 & j = m \\ 1 & m = j \pm 1 \pmod{n} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Note: We define $f(x_0) = f(x_{n-1})$ for the Periodic Laplacian.
Applying Lemma 1:

$$L^{\text{per}} f(x_j) = \sum_{m=0}^{n-1} L^{\text{per}}_{j,m} f(x_m) = \lambda f(x_j),$$

we obtain an eigenvalue of $\lambda = 2(1 - cos(\pi\alpha\delta))$.
Notice from our interval $[-1, 1]$, we have $x_0 = -1$ and $x_{n-1} = 1$. By definition we know that $f(x_0) = f(x_{n-1}) \implies e^{-i\pi\alpha} = e^{i\pi\alpha}$. This can only hold for $\alpha \in \mathbb{Z}$. When $\alpha \in \mathbb{Z}$, we see from the General Form for Eigenvalues proof that

$$Lf = \lambda f$$

as needed.

□

2.3. **Regular Laplacian.** For the Regular Laplacian we consider the function $f(x) = Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}$.

Theorem 2.2. *Eigenvalues of L are of the form $\lambda = 2(1 - \cos(\pi\alpha\delta))$.*

Proof. $L(x_j) = 1\begin{pmatrix} 0 & \dots & -1 & 2 & -1 & 0 & \dots & 0 \end{pmatrix}$ where $c = 1$. By Lemma 1 the corresponding eigenvalue is $\lambda = 2(1 - \cos(\pi\alpha\delta)$. ☐

Finding the eigenvalue is done using the internal points. We next consider the end points where $j = 0$ and $j = n - 1$ in order to solve for α, A, B in $f(x)$.

Theorem 2.3. $\alpha = \frac{c}{\delta+2}$ *where $c \in \mathbb{Z}$, $A = \pm B$ so eigenvectors are of the form $f(x) = \cos(\pi\alpha x)$ (with c even) or $\sin(\pi\alpha x)$ (with c odd).*

Proof. Consider the end points where $j = 0$ and $j = n - 1$. Then,

$$Lf(-1) = Ae^{i\pi\alpha(-1)} + Be^{-i\pi\alpha(-1)} - (Ae^{i\pi\alpha(-1+\delta)} + Be^{-i\pi\alpha(-1+\delta)}) \tag{2.2}$$

$$Lf(1) = Ae^{i\pi\alpha(1)} + Be^{-i\pi\alpha(1)} - (Ae^{i\pi\alpha(1-\delta)} + Be^{-i\pi\alpha(1-\delta)}) \tag{2.3}$$

We need $Lf(-1) = \lambda f(-1)$ and $Lf(1) = \lambda f(1)$. We can set up the system of equations to be

$$Ae^{-i\pi\alpha}(e^{-i\pi\alpha\delta} - 1) + Be^{i\pi\alpha}(e^{i\pi\alpha\delta} - 1) = 0 \tag{2.4}$$

$$Ae^{i\pi\alpha}(e^{i\pi\alpha\delta} - 1) + Be^{-i\pi\alpha}(e^{-i\pi\alpha\delta} - 1) = 0 \tag{2.5}$$

This is, equivalently,

$$e^{-i\pi\alpha}(e^{-i\pi\alpha\delta} - 1)\begin{bmatrix} 1 & -e^{i\pi\alpha(2+\delta)} \\ -e^{i\pi\alpha(2+\delta)} & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.6}$$

For $\begin{bmatrix} A \\ B \end{bmatrix}$ to be non-trivial (that is, not the zero vector), the matrix $\mathbf{M} = \begin{bmatrix} 1 & -e^{i\pi\alpha(2+\delta)} \\ -e^{i\pi\alpha(2+\delta)} & 1 \end{bmatrix}$ must be singular. Thus, we can find α when $\det(\mathbf{M}) = \mathbf{0}$. Completing this calculation gives us $\alpha = \frac{c}{\delta+2}$ where $c \in \mathbb{Z}$. When we plug in α into $\mathbf{M}$ we find that $A = (-1)^c B$. ☐

2.4. **Probabilistic Laplacian.** For the Probabilistic Laplacian we consider functions of the form

$$f(x) = Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}$$

Theorem 2.4. *Eigenvalues of $\mathcal{L}$ are of the form $\lambda = 1 - \cos(\pi\alpha\delta)$.*

Proof. For x_j with $0 < j < n - 1$, $\mathcal{L}(x_j) = \frac{1}{2}\begin{pmatrix} 0 & \dots & -1 & 2 & -1 & 0 & \dots & 0 \end{pmatrix}$ where $c = \frac{1}{2}$. By Lemma 1 the corresponding eigenvalue is $\lambda = 1 - \cos(\pi\alpha\delta)$. ☐

Finding the eigenvalue is done using the internal points (i.e. x_j with $0 < j < n-1$). We next consider the endpoints where $j = 0$ and $j = n-1$ in order to solve for α, A, B in $f(x)$.

Theorem 2.5. *$\alpha = \frac{c}{2}$ for $c \in \mathbb{Z}$, and $A = \pm B$. Thus eigenvectors of $\mathcal{L}_{n,1}$ are of the form $2A\cos(\pi\alpha x)$ (with c even) or $2Ai\sin(\pi\alpha x)$ (with c odd).*

Proof. Consider the endpoints $x_0 = -1$ and $x_{n-1} = 1$. Then we have that

$$\mathcal{L}f(-1) = Ae^{-i\pi\alpha} + Be^{i\pi\alpha} - (Ae^{i\pi\alpha(\delta-1)} + Be^{-i\pi\alpha(\delta-1)}) \tag{2.7}$$

$$\mathcal{L}f(1) = Ae^{i\pi\alpha} + Be^{-i\pi\alpha} - (Ae^{i\pi\alpha(1-\delta)} + Be^{-i\pi\alpha(1-\delta)}) \tag{2.8}$$

Since we want $\mathcal{L}f = \lambda f$, by 3.8 and 3.9, we have that

$$Ae^{-i\pi\alpha} + Be^{i\pi\alpha} - (Ae^{i\pi\alpha(\delta-1)} + Be^{-i\pi\alpha(\delta-1)}) = \frac{1}{2}(2 - e^{i\pi\alpha\delta} - e^{-i\pi\alpha\delta})(Ae^{-i\pi\alpha} + Be^{i\pi\alpha})$$
$$Ae^{i\pi\alpha} + Be^{-i\pi\alpha} - (Ae^{i\pi\alpha(1-\delta)} + Be^{-i\pi\alpha(1-\delta)}) = \frac{1}{2}(2 - e^{i\pi\alpha\delta} - e^{-i\pi\alpha\delta})(Ae^{i\pi\alpha} + Be^{-i\pi\alpha})$$

Equivalently,

$$\begin{bmatrix} -\frac{1}{2}e^{-i\pi\alpha(\delta+1)}\left(-1 + e^{2i\pi\alpha\delta}\right) & \frac{1}{2}e^{-i\pi\alpha(\delta-1)}\left(-1 + e^{2i\pi\alpha\delta}\right) \\ \frac{1}{2}e^{-i\pi\alpha(\delta-1)}\left(-1 + e^{2i\pi\alpha\delta}\right) & -\frac{1}{2}e^{-i\pi\alpha(\delta+1)}\left(-1 + e^{2i\pi\alpha\delta}\right) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Factoring out the entry along the diagonal, we have

$$-\frac{1}{2}e^{-i\pi\alpha(\delta+1)}\left(-1 + e^{2i\pi\alpha\delta}\right) \begin{bmatrix} 1 & -e^{2i\pi\alpha} \\ -e^{2i\pi\alpha} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We want to find a non-trivial solution for A, B. Then $\mathbf{M} = \begin{bmatrix} 1 & -e^{2i\pi\alpha} \\ -e^{2i\pi\alpha} & 1 \end{bmatrix}$ must be a singular matrix. So we can solve for α by fixing $\det(\mathbf{M}) = 0$. Completing this calculation yields $\alpha = \frac{c}{2}$ for $c \in \mathbb{Z}$. Using this, we can solve for $A = B$ (for even c) or $A = -B$ (for odd c). Thus, the eigenvectors of $\mathcal{L}$ are of the form $A(e^{i\pi\frac{c}{2}x} + e^{-i\pi\frac{c}{2}x})$ and $A(e^{-i\pi\frac{c}{2}} - e^{-i\pi\frac{c}{2}})$, or equivalently, $2A\cos(\pi\frac{c}{2}x)$ for even c or $2Ai\sin(\pi\frac{c}{2}x)$ for odd c. Normalizing, we find that $f(x) = \cos(\pi\frac{c}{2}x)$ if c is even and $\sin(\pi\frac{c}{2}x)$ if c is odd since A can be any scalar multiple. ☐

3. Eigencoordinates and Chebyshev Polynomials

Definition 3.1. *The Chebyshev Polynomials of the first kind T_n are defined by $T_n(\cos(\theta)) = \cos(n\theta)$.*

The following is the recursive definition for Chebyshev Polynomials of the first kind.

Definition 3.2. *The Chebyshev polynomials of the first kind are defined by the recurrence relation.*

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$\vdots$$

$$T_{\ell+1}(x) = 2xT_\ell(x) - T_{\ell-1}(x)$$

Suppose L is a Laplacian matrix and f_1 and f_ℓ are the eigenfunctions corresponding to the first and ℓ^{th} smallest eigenvalues of L. The eigencoordinates are $(f_1(x), f_\ell(x))$. We want to investigate the error, E, between our eigencoordinates and the Chebyshev polynomial T_ℓ. The error can be obtained from the following equation

$$E(x) = f_\ell(x) - T_\ell(f_1(x)). \tag{3.1}$$

Previously the input of the Chebyshev polynomials is $\cos(y)$, we want to investigate the Chebyshev polynomials with input $\sin(y)$. Following is a lemma for the formula of the $T_\ell(\sin(y))$.

Theorem 3.3.

$$T_\ell(\sin(y)) = \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}).$$

Proof. We will use induction for the proof.

Base Case: When $\ell = 0$, $T_0(\sin(y)) = \frac{1}{2}(e^0 + e^0) = 1$.

When $\ell = 1$, $T_1(\sin(y)) = \frac{1}{2i}(e^{iy} - e^{-iy}) = \frac{1}{2i}2i\sin(\ell y) = \sin(\ell y)$.

Induction: Suppose that $T_\ell(\sin(y)) = \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y})$. We want to show that $T_{\ell+1}(\sin(y)) = \frac{i^{-(\ell+1)}}{2}(e^{i(\ell+1)y} + (-1)^{\ell+1}e^{-i(\ell+1)y})$.

According to the Definition 3.2, we know that

$$\begin{aligned}
&T_{\ell+1}(\sin(y)) = 2\sin(y)T_\ell(\sin(y)) - T_{\ell-1}(\sin(y)) \\
&= 2\sin(y)\frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}) - \frac{i^{-(\ell-1)}}{2}(e^{i(\ell-1)y} + (-1)^{\ell-1}e^{-i(\ell-1)y}) \\
&= \frac{i^{-(l+1)}}{2}(2\frac{(e^{iy} - e^{-iy})}{2i}i(e^{i\ell y} + (-1)^\ell e^{-i\ell y}) - i^2(e^{i(\ell-1)y} + (-1)^{\ell-1}e^{-i(\ell-1)y})) \\
&= \frac{i^{-(l+1)}}{2}((e^{iy} - e^{-iy})(e^{i\ell y} + (-1)^\ell e^{-i\ell y}) + (e^{i(\ell-1)y} + (-1)^{\ell-1}e^{-i(\ell-1)y})) \\
&= \frac{i^{-(l+1)}}{2}(e^{i(\ell+1)y} + (-1)^\ell e^{-i(\ell-1)y} - e^{i(\ell-1)y} + (-1)^{\ell+1}e^{-i(\ell+1)y} + e^{i(\ell-1)y} + (-1)^{\ell-1}e^{-i(\ell-1)y}) \\
&= \frac{i^{-(\ell+1)}}{2}(e^{i(\ell+1)y} + (-1)^{\ell+1}e^{-i(\ell+1)y})
\end{aligned}$$

Thus, we can conclude that

$$T_\ell(\sin(y)) = \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}).$$

□

The following are the first few Chebyshev polynomials of $\sin(y)$

$$\begin{aligned}
T_0(\sin(y)) &= 1\\
T_1(\sin(y)) &= \sin(y)\\
T_2(\sin(y)) &= -\cos(2y)\\
T_3(\sin(y)) &= -\sin(3y)\\
T_4(\sin(y)) &= \cos(4y)\\
T_5(\sin(y)) &= \sin(5y)\\
&\vdots
\end{aligned}$$

3.1. **Regular Laplacian and Chebyshev Polynomial.** Suppose L is a regular Laplacian matrix with $k = 1$. We want to investigate the correlation between eigencoordinates and $T_\ell(\sin(y))$. According to Theorem 2.3, we know the eigenfunctions of L are $\cos(c\pi\frac{n-1}{2n}x)$ if c is even and $\sin(c\pi\frac{n-1}{2n}x)$ if c is odd, where c is an integer from 0 to $n-1$.

Corollary 3.4. *$E(x) = f_\ell(x) - T_\ell(f_1(x)) = 0$ for the regular Laplacian matrix.*

Proof. Suppose ℓ is even. Then, $f_\ell(x) = \cos(\ell\pi\frac{n-1}{2n}x)$. Let $y = \pi\frac{n-1}{2n}x$. We also know $-\cos(\ell y)$ is an eigenfunction corresponding to the ℓ^{th} smallest eigenvalue because it is a scalar multiple of $f_\ell(x)$. Notice that $f_1(x) = \sin(\pi\frac{n-1}{2n}x) = \sin(y)$. Then, we have

$$E(x) = f_\ell(x) - T_\ell(\sin(y))$$

According to the lemma 3.3,

$$
\begin{aligned}
E(x) &= f_\ell(x) - T_\ell(\sin(y)) \\
&= f_\ell(x) - \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}) \\
&= \begin{cases} f_\ell(x) - \frac{1}{2}(e^{i\ell y} + e^{-i\ell y}), & \text{if } \ell \bmod 4 = 0 \\ f_\ell(x) + \frac{1}{2}(e^{i\ell y} + e^{-i\ell y}), & \text{if } \ell \bmod 4 = 2 \end{cases} \\
&= \begin{cases} f_\ell(x) - \cos(\ell y), & \text{if } \ell \bmod 4 = 0 \\ f_\ell(x) + \cos(\ell y), & \text{if } \ell \bmod 4 = 2 \end{cases}
\end{aligned}
$$

Since f_ℓ can be either $\pm\cos(\ell y)$, we can conclude that $E(x) = f_\ell(x) - T_\ell(\sin(y)) = 0$.

Suppose ℓ is odd. Then, $f_\ell(x) = \sin(\ell\pi\frac{n-1}{2n}x)$. Let $y = \pi\frac{n-1}{2n}x$. We also know $-\sin(\ell y)$ is an eigenfunction corresponding to the ℓ^{th} smallest eigenvalue because it is a scalar multiple of $f_\ell(x)$. Notice that $f_1(x) = \sin(\pi\frac{n-1}{2n}x) = \sin(y)$. Then, we have

$$E(x) = f_\ell(x) - T_\ell(\sin(y))$$

According to the Theorem 3.3,

$$
\begin{aligned}
E(x) &= f_\ell(x) - T_\ell(\sin(y)) \\
&= f_\ell(x) - \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}) \\
&= \begin{cases} f_\ell(x) - \frac{1}{2i}(e^{i\ell y} - e^{-i\ell y}), & \text{if } \ell \bmod 4 = 1 \\ f_\ell(x) + \frac{1}{2i}(e^{i\ell y} + e^{-i\ell y}), & \text{if } \ell \bmod 4 = 3 \end{cases} \\
&= \begin{cases} f_\ell(x) - \sin(\ell y), & \text{if } \ell \bmod 4 = 1 \\ f_\ell(x) + \sin(\ell y), & \text{if } \ell \bmod 4 = 3 \end{cases}
\end{aligned}
$$

Since f_ℓ can be either $\pm\sin(\ell y)$, we can conclude that $E(x) = f_\ell(x) - T_\ell(\sin(y)) = 0$. □

3.2. **Probabilistic Laplacian and Chebyshev Polynomial.** Suppose $\mathcal{L}$ is a probabilistic Laplacian matrix with $k = 1$. We want to investigate the correlation between eigencoordinates and $T_\ell(\sin(y))$. According to Theorem 2.5, we know the eigenfunctions of $L_{n,1}$ are $\cos(\frac{c\pi x}{2})$ if c is even and $\sin(\frac{c\pi x}{2})$ if c is odd, where c is an integer from 0 to $n-1$.

Corollary 3.5. $E(x) = f_\ell(x) - T_\ell(f_1(x)) = 0$ *for the probabilistic Laplacian matrix.*

Proof. Suppose ℓ is even. Then, $f_\ell(x) = \cos(\frac{\ell\pi x}{2})$. Let $y = \frac{\pi x}{2}$. We also know $-cos(\ell y)$ is an eigenfunction corresponding to the ℓ^{th} smallest eigenvalue because it is a scalar multiple of $f_\ell(x)$. Notice that $f_1(x) = \sin(\frac{\pi x}{2}) = \sin(y)$. Then, we have

$$E(x) = f_\ell(x) - T_\ell(\sin(y))$$

According to the Theorem 3.3,

$$\begin{aligned} E(x) &= f_\ell(x) - T_\ell(\sin(y)) \\ &= f_\ell(x) - \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}) \\ &= f_\ell(x) \pm \frac{1}{2}(e^{i\ell y} + e^{-i\ell y}) \\ &= \begin{cases} f_\ell(x) - \cos(\ell y), & \text{if } \ell \bmod 4 = 0 \\ f_\ell(x) + \cos(\ell y), & \text{if } \ell \bmod 4 = 2 \end{cases} \end{aligned}$$

Since f_ℓ can be either $\pm\cos(\ell y)$, we can conclude that $E(x) = f_\ell(x) - T_\ell(\sin(y)) = 0$.

Suppose ℓ is odd. Then, $f_\ell(x) = \sin(\frac{\ell\pi x}{2})$. Let $y = \frac{\pi x}{2}$. We also know $-\sin(\ell y)$ is an eigenfunction corresponding to the ℓ^{th} smallest eigenvalue because it is a scalar multiple of $f_\ell(x)$. Notice that $f_1(x) = \sin(\frac{\pi x}{2}) = \sin(y)$. Then, we have

$$E(x) = f_\ell(x) - T_\ell(\sin(y))$$

According to the Theorem 3.3,

$$\begin{aligned} E(x) &= f_\ell(x) - T_\ell(\sin(y)) \\ &= f_\ell(x) - \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}) \\ &= \begin{cases} f_\ell(x) - \frac{1}{2i}(e^{i\ell y} - e^{-i\ell y}), & \text{if } \ell \bmod 4 = 1 \\ f_\ell(x) + \frac{1}{2i}(e^{i\ell y} + e^{-i\ell y}), & \text{if } \ell \bmod 4 = 3 \end{cases} \\ &= \begin{cases} f_\ell(x) - \sin(\ell y), & \text{if } \ell \bmod 4 = 1 \\ f_\ell(x) + \sin(\ell y), & \text{if } \ell \bmod 4 = 3 \end{cases} \end{aligned}$$

Since f_ℓ can be either $\pm\sin(\ell y)$, we can conclude that $E(x) = f_\ell(x) - T_\ell(\sin(y)) = 0$. ☐

3.3. **Periodic Laplacian and Chebyshev Polynomial.**

Theorem 3.6. *Assume that $f_1(x)$ is the eigenfunction corresponding to the 1^{st} non-zero eigenvalue λ_1 of L^{per} and $f_\ell(x)$ is the ℓ^{th} eigenfunction corresponding to the ℓ^{th} eigenvalue λ_ℓ of L^{per}, $f_\ell(x) = T_\ell(x)$.*

Proof. From Theorem 2.1 an eigenvector of L^{per} is of the form $e^{i\pi\ell x}$. λ_ℓ has multiplicity 2 and the corresponding eigenvectors are $\{e^{i\pi\ell x}, e^{-i\pi\ell x}\}$ which is a basis for the eigenspace of λ_ℓ. Let $y = \pi x$. So, $f_\ell(x) = \frac{e^{i\ell y}+e^{-i\ell y}}{2} = \cos(\ell y)$ is also an eigenvector for the eigenspace of λ_ℓ. Thus, $\frac{e^{iy}+e^{-iy}}{2} = \cos(y)$ is an eigenvector for the eigenspace of λ_1.

$$\begin{aligned} E(x) &= f_\ell(x) - T_\ell(f_1(x)) \\ &= f_\ell(x) - T_\ell(\cos(y)) \\ &= f_\ell(x) - \cos(\ell y) = 0 \end{aligned}$$

□

4. Analysis of Robin Problem on $[-1, 1]$

4.1. Continuous Robin Problem with Uniformly Spaced Points.

Definition 4.1. *Suppose u is an eigenfunction of the Laplacian $L_{n\to\infty,k=1}$ with corresponding eigenvalue λ. The Robin Boundary Conditions are*

$$\begin{cases} u'' = -\lambda u & \text{on } (-1,1) \\ \partial_n u = \rho u & \text{at } x = \pm 1 \end{cases} \tag{4.1}$$

where ∂_n is the outward normal derivative at $x = \pm 1$.

Solving the differential equation $u'' = -\lambda u$ gives the general solution for u, $u(x) = Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}$. Thus, $\partial_n u(x) = sgn(x) i\pi\alpha(Ae^{i\pi\alpha x} - Be^{-i\pi\alpha x})$ at $x = \pm 1$.

For the boundary equations, when $x = -1$,

$$\begin{aligned} 0 = \partial_n u(-1) - \rho u(-1) &= (-1)i\pi\alpha(Ae^{-i\pi\alpha} - Be^{i\pi\alpha}) - \rho(Ae^{-i\pi\alpha} + Be^{i\pi\alpha}) \\ &= (-1)(e^{-i\pi\alpha}(i\pi\alpha + \rho)A - e^{i\pi\alpha}(i\pi\alpha - \rho)B). \end{aligned}$$

When $x = 1$,

$$\begin{aligned} 0 = \partial_n u(1) - \rho u(1) &= i\pi\alpha(Ae^{i\pi\alpha} - Be^{-i\pi\alpha}) - \rho(Ae^{i\pi\alpha} + Be^{-i\pi\alpha}) \\ &= e^{i\pi\alpha}(i\pi\alpha - \rho)A - e^{-i\pi\alpha}(i\pi\alpha + \rho)B. \end{aligned}$$

This is equivalent,

$$\begin{bmatrix} e^{-i\pi\alpha}(i\pi\alpha + \rho) & -e^{i\pi\alpha}(i\pi\alpha - \rho) \\ e^{i\pi\alpha}(i\pi\alpha - \rho) & -e^{-i\pi\alpha}(i\pi\alpha + \rho) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.2}$$

To find nontrivial values for A and B, the determinant of the matrix need to be 0.

$$\det \begin{bmatrix} e^{-i\pi\alpha}(i\pi\alpha+\rho) & -e^{i\pi\alpha}(i\pi\alpha-\rho) \\ e^{i\pi\alpha}(i\pi\alpha-\rho) & -e^{-i\pi\alpha}(i\pi\alpha+\rho). \end{bmatrix}$$
$$= -e^{-2i\pi\alpha}(i\pi\alpha+\rho)^2 + e^{2i\pi\alpha}(i\pi\alpha-\rho)^2 = 0.$$

After some simplification,

$$e^{2i\pi\alpha} = \pm\frac{i\pi\alpha+\rho}{i\pi\alpha-\rho}. \tag{4.3}$$

Substituting $i\pi\alpha+\rho = e^{2i\pi\alpha}(i\pi\alpha-\rho)$ into the matrix 4.2, we obtain the relationship for A and B: $A = B$. Similarly, substituting $i\pi\alpha+\rho = -e^{2i\pi\alpha}(i\pi\alpha-\rho)$ into the matrix 4.2, we arrive at the relationship for A and B: $A = -B$.

Then, the general equation for u is $u = Ae^{i\pi\alpha x} \pm Ae^{-i\pi\alpha x}$. Since A can be any scalar multiple,

$$u = \cos(\pi\alpha x) \text{ or } u = \sin(\pi\alpha x)$$

with value of α from the equation 4.3.

For the equation 4.3, we have a formula for ρ in terms of α,

$$\rho(\alpha) = i\pi\alpha\frac{(\pm e^{2i\pi\alpha}-1)}{(\pm e^{2i\pi\alpha}+1)} = i\pi\alpha\frac{(\pm e^{i\pi\alpha}-e^{-i\pi\alpha})}{(\pm e^{i\pi\alpha}+e^{-i\pi\alpha})} = \begin{cases} -\pi\alpha\tan(\pi\alpha), & \text{when } u = \cos(\pi\alpha x) \\ \pi\alpha\cot(\pi\alpha), & \text{when } u = \sin(\pi\alpha x) \end{cases}. \tag{4.4}$$

(1) $\rho(\alpha) = 0$.

$-\pi\alpha\tan(\pi\alpha) = 0$ if and only if $\alpha = 0$ or $\tan(\pi\alpha) = 0$ if and only if $\alpha \in \mathbb{Z}$. And, $\pi\alpha\cot(\pi\alpha) = 0$ if and only if $\alpha = 0$ or $\cot(\pi\alpha) = 0$ if and only if $\alpha \in \mathbb{Z} + \frac{1}{2}$. Therefore, $\rho(\alpha) = 0$ if and only if $\alpha \in \frac{c}{2}$, where $c \in \mathbb{Z}$. Thus,

$$u = \begin{cases} \sin(\pi\frac{c}{2}x), & \text{when } c \text{ is odd} \\ \cos(\pi\frac{c}{2}x), & \text{when } c \text{ is even} \end{cases}.$$

If $c \geq 1$, then we have the eigenvectors,

$$u_1(x) = \sin(\frac{1}{2}\pi x)$$
$$u_2(x) = \cos(\pi x)$$
$$u_3(x) = \sin(\frac{3}{2}\pi x)$$
$$u_4(x) = \cos(2\pi x)$$
$$\vdots$$

By Theorem 3.3, we know $(u_1(x), u_j(x))$ forms Chebyshev polynomials of the first kind.

4.2. Discrete Robin Problem with Uniformly Spaced Points.

Definition 4.2. *Suppose u is an eigenfunction of the Laplacian matrix $L_{n,k=1}$ of any variety with corresponding eigenvalue λ. The Discrete Robin Boundary Conditions are*

$$\begin{cases} u'' = -\lambda u & \text{on } (-1,1) \\ Lu = \sigma u & \text{at } x = \pm 1 \end{cases}. \tag{4.5}$$

We want to see how σ is related to ρ in the Robin Boundary Conditions 4.5.

4.3. Discrete Robin Problem on the Probabilistic Laplacian.

Definition 4.3. *The Robin Boundary Conditions for the Probabilistic Laplacian are*

$$\begin{cases} u'' = -\lambda u & \text{on } (-1,1) \\ L_{prob} u = \sigma u & \text{at } x = \pm 1 \end{cases}. \tag{4.6}$$

Theorem 4.4. *When $n \to \infty$, $\sigma \to 0$.*

Proof. $u(x) = Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}$ is the general form for eigenvectors of probabilistic Laplacian matrices, L_{prob}. Suppose that the first row of L_{prob} is

$$L_{prob}(x_0) = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \end{pmatrix}$$

, and the last row is

$$L_{prob}(x_{n-1}) = \begin{pmatrix} 0 & \dots & 0 & -1 & 1 \end{pmatrix}$$

For the boundary equations, when $x = -1$,

$$\begin{aligned} 0 = L_{prob}u(-1) - \sigma u(-1) &= Ae^{-i\pi\alpha} + Be^{i\pi\alpha} - (Ae^{i\pi\alpha(-1+\delta)} + Be^{-i\pi\alpha(-1+\delta)}) - \sigma(Ae^{-i\pi\alpha} + Be^{i\pi\alpha}) \\ &= ((1-\sigma)e^{-i\pi\alpha} - e^{i\pi\alpha(-1+\delta)})A + ((1-\sigma)e^{i\pi\alpha} - e^{-i\pi\alpha(-1+\delta)})B \end{aligned}$$

When $x = 1$,

$$\begin{aligned} 0 = L_{prob}u(1) - \sigma u(1) &= Ae^{i\pi\alpha} + Be^{-i\pi\alpha} - (Ae^{i\pi\alpha(1-\delta)} + Be^{-i\pi\alpha(1-\delta)}) - \sigma(Ae^{i\pi\alpha} + Be^{-i\pi\alpha}) \\ &= ((1-\sigma)e^{i\pi\alpha} - e^{i\pi\alpha(1-\delta)})A + ((1-\sigma)e^{-i\pi\alpha} - e^{-i\pi\alpha(1-\delta)})B \end{aligned}$$

This is equivalent,

$$\begin{bmatrix} (1-\sigma)e^{-i\pi\alpha} - e^{i\pi\alpha(-1+\delta)} & +(1-\sigma)e^{i\pi\alpha} - e^{-i\pi\alpha(-1+\delta)} \\ (1-\sigma)e^{i\pi\alpha} - e^{i\pi\alpha(1-\delta)} & +(1-\sigma)e^{-i\pi\alpha} - e^{-i\pi\alpha(1-\delta)} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.7}$$

To find nontrivial values for A and B, the determinant of the matrix need to be 0.

$$\det \begin{bmatrix} (1-\sigma)e^{-i\pi\alpha} - e^{i\pi\alpha(-1+\delta)} & +(1-\sigma)e^{i\pi\alpha} - e^{-i\pi\alpha(-1+\delta)} \\ (1-\sigma)e^{i\pi\alpha} - e^{i\pi\alpha(1-\delta)} & +(1-\sigma)e^{-i\pi\alpha} - e^{-i\pi\alpha(1-\delta)} \end{bmatrix}$$
$$= ((1-\sigma)e^{-i\pi\alpha} - e^{-i\pi\alpha(1-\delta)})^2 - ((1-\sigma)e^{i\pi\alpha} - e^{i\pi\alpha(1-\delta)})^2 = 0.$$

After some simplification, we have $(1-\sigma)e^{-i\pi\alpha} - e^{-i\pi\alpha(1-\delta)} = \pm((1-\sigma)e^{i\pi\alpha} - e^{i\pi\alpha(1-\delta)})$

(1) If $(1-\sigma)e^{-i\pi\alpha} - e^{-i\pi\alpha(1-\delta)} = (1-\sigma)e^{i\pi\alpha} - e^{i\pi\alpha(1-\delta)}$, then

$$0 = -(1-\sigma)(e^{i\pi\alpha} - e^{-i\pi\alpha}) + (e^{i\pi\alpha(1-\delta)} - e^{-i\pi\alpha(1-\delta)})$$
$$= -(1-\sigma)2i\sin(\pi\alpha) + 2i\sin(\pi\alpha(1-\delta)).$$

Completing the calculation, we get

$$\lim_{n\to\infty} \sigma = \lim_{n\to\infty} 1 - \frac{\sin(\pi\alpha(1-\delta))}{\sin(\pi\alpha)} = \lim_{n\to\infty} 1 - \frac{\sin(\pi\alpha)}{\sin(\pi\alpha)} = \lim_{n\to\infty} 1 - 1 = 0.$$

(2) If $(1-\sigma)e^{-i\pi\alpha} - e^{-i\pi\alpha(1-\delta)} = -((1-\sigma)e^{i\pi\alpha} - e^{i\pi\alpha(1-\delta)})$, then

$$0 = (1-\sigma)(e^{i\pi\alpha} + e^{-i\pi\alpha}) - (e^{i\pi\alpha(1-\delta)} + e^{-i\pi\alpha(1-\delta)})$$
$$= (1-\sigma)2i\cos(\pi\alpha) - 2i\cos(\pi\alpha(1-\delta)).$$

Completing the calculation, we get

$$\lim_{n\to\infty} \sigma = \lim_{n\to\infty} 1 - \frac{\cos(\pi\alpha(1-\delta))}{\cos(\pi\alpha)} = \lim_{n\to\infty} 1 - \frac{\cos(\pi\alpha)}{\cos(\pi\alpha)} = \lim_{n\to\infty} 1 - 1 = 0.$$

□

Corollary 4.5. *The eigencoordinates for the L_{prob} are the Chebyshev polynomials of the first kind, T_n.*

Proof. Previous work on the continuous Laplacian shows that when $\rho = 0$, the eigencoordinates will be exactly the Chebyshev polynomials. Hence, $\lim_{n\to\infty} \sigma = 0 = \rho$ implies that the eigencoordinates for the Probabilistic Laplacian matrix are the Chebyshev polynomials. □

4.4. **Discrete Robin Problem on Regular Laplacian.**

Corollary 4.6. *The eigencoordinates for the L_{reg} are the Chebyshev polynomials of the first kind, T_n.*

Proof. Since the boundaries of the Regular and Probabilistic Laplacians are the same we get the same result. $\lim_{n\to\infty} \sigma = 0$ implies that the eigencoordinates for the Regular Laplacian matrix are the Chebyshev polynomials. □