Representations of Indifference Prices on a Finite Probability Space

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Intuition of Utility Maximization

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A **derivative security** pays some amount at expiry depending on the value of its underlying asset. An example is a call option.

How can we price derivative securities??

Replication pricing vs Indifference pricing

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Under complete markets, derivative securities are replicable so we show the unique no-arbitrage price of our derivative security is the risk-neutral expectation of the derivative security's payoff. But this isn't the case for incomplete markets. What now?

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Intuition of Indifference Pricing

An **indifference price** is a price at which the investor gets the same expected utility level by trading a derivative security as he would if he chooses not to trade the derivative security.

Mathematical representation of indifference pricing

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We defined the value function:

$$u(x) = \sup_{X_T \in C(x)} E^{\mathbb{P}}[U(X_T)]$$

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New value function

We will define a new value function V that takes into account the number of derivative security contracts we want to take part in:

$$V(x - q\Pi, q) = \sup_{X_T \in C(x-q\Pi)} E^{\mathbb{P}}[U(X_T + qf)] \text{ for some } q \in \mathbb{R}$$

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We represent the indifference price Π of a derivative security when the following is true:

$$V(x - q\Pi, q) = V(x, 0) = u(x)$$

Indifference Price: Main Result

Theorem [F., H., Mostovyi - '23]

Let $x \in dom(u)$ be fixed and interest rate $r = 0$. Then for every derivative security f, the set of indifference prices at x is a singleton and with $y = u'(x)$, we have:

$$\Pi(x) = \mathbb{E}^{\mathbb{P}}\left[\frac{\hat{Y}_T(y)}{y} f\right] = \mathbb{E}^{\hat{\mathbb{Q}}(y)}[f]$$

Where $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \frac{\hat{Y}_T(y)}{y}$ and $\hat{\mathbb{Q}}$ is the dual optimal martingale measure from the utility maximization problem.

Proving Π is an indifference price

Check that $\Pi(x) = \mathbb{E}^{\mathbb{P}}\left[\frac{\hat{Y}_T(y)}{y} f\right] = \mathbb{E}^{\hat{\mathbb{Q}}(y)}[f]$ is an indifference price:

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Suppose we buy $q \in \mathbb{R}$ contracts, then we have $X_T \in \mathcal{X}(x - q\Pi)$. It follows that $\mathbb{E}^{\hat{\mathbb{Q}}}\left[X_T\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\hat{Y}_T(y)}{y} X_T\right] = x - q\Pi$. Then:

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$$\begin{aligned} V(x - q\Pi, q) &= \mathbb{E}^{\mathbb{P}}[U(X_T + qf)] \\ &= \mathbb{E}^{\mathbb{P}}[U(X_T + qf)] + \sum_{k=1}^{K} \lambda_k (\mathbb{E}^{\mathbb{Q}^k}[X_T] - (x - q\Pi)) \end{aligned}$$

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V(x - q\Pi, q) &= \mathbb{E}^{\mathbb{P}}[U(X_T + qf)] \\
&= \mathbb{E}^{\mathbb{P}}[U(X_T + qf)] + \sum_{k=1}^{K} \lambda_k (\mathbb{E}^{\mathbb{Q}^k}[X_T] - (x - q\Pi))
\end{aligned}
$$

Simplify Lagrangian

- Let $y = \sum_{k=1}^{K} \lambda_k$ and $\mu_k = \frac{\lambda_k}{y} \implies \mathbb{Q} = \sum_{k=1}^{K} \mu_k \mathbb{Q}^k$
- Notice $\mathbb{Q} \in \mathcal{M}^{\alpha}(S)$ since it is a convex combination of points in $\mathcal{M}^{\alpha}(S)$

Proving Π is an indifference price

Simplify Lagrangian:

$$\begin{aligned} L(X_T, y, \mathbb{Q}) &= \mathbb{E}^{\mathbb{P}}\left[U(X_T + qf)\right] + \sum_{k=1}^{K} \lambda_k (\mathbb{E}^{\mathbb{Q}^k}\left[X_T\right] - (x - q\Pi)) \\ &= \mathbb{E}^{\mathbb{P}}\left[U(X_T)\right] - y\mathbb{E}^{\mathbb{Q}}\left[X_T\right] + y(x - q\Pi) \\ &= \mathbb{E}^{\mathbb{P}}\left[U(X_T) - y\frac{\hat{Y}_T(y)}{y}X_T\right] + y(x - q\Pi) \\ &= \mathbb{E}^{\mathbb{P}}\left[V(\hat{Y}_T)\right] + y\mathbb{E}^{\mathbb{Q}}\left[X_T + qf\right] = \mathbb{E}^{\mathbb{P}}\left[V(\hat{Y}_T) + \hat{Y}_T(X_T + qf)\right] \end{aligned}$$

Proving Π is an indifference price

$$\begin{aligned} V(x - q\Pi, q) &= \mathbb{E}^{\mathbb{P}}\left[U(X_T + qf)\right] = L(X_T, y, \mathbb{Q}) \\ &= \mathbb{E}^{\mathbb{P}}\left[V(\hat{Y}_T) + \hat{Y}_T(X_T + qf)\right] \\ &= v(y) + \mathbb{E}^{\mathbb{P}}\left[\hat{Y}_T X_T\right] + \mathbb{E}^{\mathbb{P}}\left[\hat{Y}_T qf\right] \\ &= v(y) + (x - q\Pi)y + q\Pi y = u(x) = V(x, 0) \end{aligned}$$

We note that we can show this indifference price is unique under some initial wealth and some initial number of contracts bought.

Useful Results

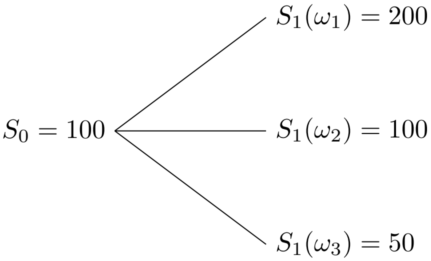
- $x = -v'(y)$
- $v(y) = \min_{\mathbb{Q} \in \mathcal{M}^{\alpha}(S)} E^{\mathbb{P}}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})] = E^{\mathbb{P}}[V(y\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}})]$
- $v'(y) = E^{\hat{\mathbb{Q}}}[V'(y\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}})]$
- $U'(X_N) = y\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$
- $\Pi(x) = \mathbb{E}^{\hat{\mathbb{Q}}(y)}\left[f\right]$

Probability Notation

- $\mathbb{P}(\omega_n) = p_n$
- $\mathbb{Q}(\omega_n) = q_n$
- $\hat{\mathbb{Q}}(\omega_n) = \hat{q}_n$

Trinomial Model Example

- Initial market conditions: $S_0 = 100, \ r = 0$.
- Probability space: $\Omega = \{\omega_1, \omega_2, \omega_3\}$
- Actual Probabilities: $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.5$
- $x = 5$, $U(x) = \ln(x)$



Trinomial Model Example

- $S_0 = \frac{q_1 S_1(\omega_1) + q_2 S(\omega_2) + q_3 S_1(\omega_3)}{1+r}$
- Sum of all probabilities add to 1

Trinomial Model Example

$$200q_1 + 100q_2 + 50q_3 = 100$$
$$q_1 + q_2 + q_3 = 1$$

- $q_1 = t$, $q_2 = 1 - 3t$, $q_3 = 2t$, $t \in (0, \frac{1}{3})$

Trinomial Model Example

Conjugate: $V(y) = -\ln(y) - 1$

$$V'(y) = -\frac{1}{y}$$

$$v(y) = \min_{\mathbb{Q}\in\mathcal{M}^{\alpha}(S)} E^{\mathbb{P}}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})]$$

Trinomial Model Example

$$v(y) = \min_{\mathbb{Q}\in\mathcal{M}^\alpha(S)} E^{\mathbb{P}}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})]$$

$$= \min_{\mathbb{Q}\in\mathcal{M}^\alpha(S)} \Sigma_{n=1}^{3}\, p_n(-\ln(y\frac{q_n}{p_n}) - 1)$$

$$= \min_{\mathbb{Q}\in\mathcal{M}^\alpha(S)} 0.2(-\ln(y\frac{q_1}{0.2}) - 1) + 0.3(-\ln(y\frac{q_2}{0.3}) - 1) + 0.5(-\ln(y\frac{q_3}{0.5}) - 1)$$

$$= \min_{\mathbb{Q}\in\mathcal{M}^\alpha(S)} -\ln(y) - 0.2\ln(\frac{t}{0.2}) - 0.3\ln(\frac{1-3t}{0.3}) - 0.5\ln(\frac{2t}{0.5}) - 1$$

Trinomial Model Example

$$\frac{d}{dt} \to 0.2(\frac{1}{\frac{t}{0.2}} \cdot \frac{1}{0.2}) + 0.3(\frac{1}{\frac{1-3t}{0.3}} \cdot -\frac{3}{0.3}) + 0.5(\frac{1}{\frac{2t}{0.5}} \cdot \frac{2}{0.5})$$

$$= \frac{0.2}{t} - \frac{0.9}{1-3t} + \frac{1}{2t}$$

$$= \frac{0.7}{t} - \frac{0.9}{1-3t}$$

$$= \frac{0.7(1-3t) - 0.9(t)}{t(1-3t)}$$

$$= \frac{0.7-3t}{t(1-3t)} \Rightarrow 0$$

Trinomial Model Example

$$0.7 - 3t = 0$$

$$t = \frac{7}{30}$$

$$\hat{\mathbb{Q}} = \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \end{pmatrix} = \begin{pmatrix} \frac{7}{30} \\ \frac{9}{30} \\ \frac{14}{30} \end{pmatrix}$$

Trinomial Model Example

$$x = -v'(y) = -(E^{\hat{\mathbb{Q}}}[V'(y\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}})])$$

$$= -\Sigma_{n=1}^{3}\hat{q}_n \cdot (-\frac{1}{y(\frac{\hat{q}_n}{p_n})}) = \Sigma_{n=1}^{3}\frac{p_n}{y}$$

$$5 = \frac{0.2}{y} + \frac{0.3}{y} + \frac{0.5}{y} = \frac{1}{y}$$

$$y = 0.20$$

Trinomial Model Example

$$U'(X_N) = y\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$$

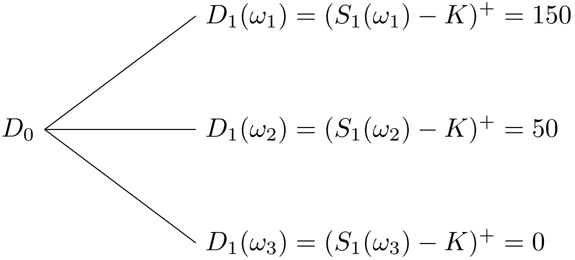
$$U'(X_N(\omega_n)) = y\frac{\hat{q}_n}{p_n};\ n = 1, 2, 3$$

$$X_N(\omega_n) = \frac{1}{y\frac{\hat{q}_n}{p_n}} = \frac{p_n}{y \cdot \hat{q}_n}$$

$$\begin{pmatrix} X_N(\omega_1) \\ X_N(\omega_2) \\ X_N(\omega_3) \end{pmatrix} = \begin{pmatrix} \frac{p_1}{y\cdot\hat{q}_1} \\ \frac{p_2}{y\cdot\hat{q}_2} \\ \frac{p_3}{y\cdot\hat{q}_3} \end{pmatrix} = \begin{pmatrix} \frac{0.2}{(0.2)(\frac{7}{30})} \\ \frac{0.3}{(0.2)(\frac{9}{30})} \\ \frac{0.5}{(0.2)(\frac{14}{30})} \end{pmatrix} = \begin{pmatrix} 4.29 \\ 5.00 \\ 5.36 \end{pmatrix}$$

Trinomial Model Example

- D: Price of Derivative Security
- Strike Price: $K = 50$



Trinomial Model Example

$$\Pi(x) = E_{\hat{\mathbb{Q}}}[f]$$

$$\Pi(5) = \mathbb{E}_{\hat{\mathbb{Q}}}[D]$$

$$= \sum_{i=1}^{3} D_1(\omega_i) \cdot \hat{q}_i$$

$$= 150 \cdot \frac{7}{30} + 50 \cdot \frac{9}{30} + 0 \cdot \frac{14}{30}$$

$$= 50$$

Next Steps

Under markets with interest rate, we can apply a change of numeraire to essentially convert the problem to one without interest (the discount factor is an example of changing the numeraire).