Approximation of Laplacian Operators

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Outline

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$\mathcal{L}_{n,\varepsilon} f(x)$: probabilistic Laplacian of f depending on n and ε

$\mathcal{L}_{\varepsilon} f(x) = \frac{1}{|\mathcal{B}(x,\varepsilon)|} \int_{\mathcal{B}(x,\varepsilon)} (f(x) - f(y))dy$

Giné and Koltchinskii (2006)

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For $f \in C^3$, they approximate the Laplace-Beltrami operator using a weighted averaging operator,

$$\Delta_{h_n,n} f(p) := \frac{1}{n h_n^{d+2}} \sum_{i=1}^{n} (f(X_i) - f(p)) \cdot K(\frac{p - X_i}{h_n}), p \in M$$

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$K(\frac{p-X_i}{h_n})$ is the Gaussian kernel and $h_n \to 0$ as $n \to \infty$.

h_n pertains to our usage of ε.

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$$\mathcal{L}_\varepsilon f(x) \to Lf(x) \text{ as } \varepsilon \to 0$$

$$|\mathcal{L}_\varepsilon - \mathcal{L}_{n,\varepsilon}| = |L - \frac{1}{\varepsilon^2}\mathcal{L}_\varepsilon| + |\frac{1}{\varepsilon^2}\mathcal{L}_\varepsilon - \frac{1}{\varepsilon^2}\mathcal{L}_{n,\varepsilon}|$$

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Let X be an arbitrary metric space. There is no guarantee of the existence of $(+, \cdot)$ in X, and thus we cannot form the derivatives necessary for a Taylor series in X.

Instead, let us consider a sufficiently smooth function f on $[-1, 1]$. Then on the interval $x_m + \varepsilon, x_m - \varepsilon$ we have the Taylor series expansion

$$f(y) = f(x_m) + f'(x_m)(y - x_m) + \frac{f''(x_m)(y - x_m)^2}{2} + ...$$

Utilizing Symmetry

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Fix $\varepsilon \in (0, 1)$. For f on $[-1, 1]$, define a function $\tilde{f}$ on $[-1-\varepsilon, 1+\varepsilon]$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-1, 1] \\ f(2-x) & x \in (1, 1+\varepsilon] \\ f(-2-x) & x \in [-1-\varepsilon, -1) \end{cases} \tag{2.1}$$

Utilizing Symmetry

Lemma (1)

If $f \in C^2[-1,1]$, i.e. if both f' and f'' both exist and are both continuous, and $f'(1) = f'(-1) = 0$, then $\tilde{f} \in C^2$.

Utilizing Symmetry

Lemma (1)

If $f \in C^2[-1,1]$*, i.e. if both* f' *and* f'' *both exist and are both continuous, and* $f'(1) = f'(-1) = 0$*, then* $\tilde{f} \in C^2$*.*

Theorem (1)

If $f \in C^2[-1,1]$ *then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{L}_\varepsilon f(x) = -\frac{f''(x)}{6}.$$

Proof of Theorem (1)

Recall from Lemma 1 that since $f'(1) = f'(-1) = 0, \tilde{f}$ is sufficiently smooth so that we may look at the Taylor series. If $x \in [-1, 1]$ then take $x < min(|x-1|, |x+1|)$. Consider the Taylor expansion of $\tilde{f}$,

$$\tilde{f}(y) = \tilde{f}(x) + \tilde{f}'(x)(y-x) + \frac{\tilde{f}''(x)}{2}(y-x)^2 + o(|y-x|^2)$$

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Using some algebra and basic integration, we have

$$\int_{x-\varepsilon}^{x+\varepsilon} \tilde{f}(x) - \tilde{f}(y)dy = -\tilde{f}'(x)\int_{x-\varepsilon}^{x+\varepsilon}(y-x)dy - \frac{\tilde{f''}(x)}{2}\int_{x-\varepsilon}^{x+\varepsilon}(y-x)^2dy + \int_{x-\varepsilon}^{x+\varepsilon} o(|y-x|^2)dy$$

Proof of Theorem (1) cont'd.

We will use a substition $t = y - x$ and algebraic manipulation to obtain

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} \tilde{f}(x) - \tilde{f}(y)dy = -\frac{\tilde{f}''(x)\varepsilon^2}{6} + \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} t^2 dt \cdot o(\varepsilon^2)$$

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Observe that the following inequality holds;

$$|\mathcal{L}_\varepsilon f(x) + \frac{\tilde{f}''(x)e^2}{6}| \leq \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} t^2 dt \cdot o(\varepsilon^2) = \frac{\varepsilon^2}{3} \cdot o(\varepsilon^2)$$

So,

$$|\frac{1}{\varepsilon^2}\mathcal{L}_\varepsilon f(x) + \frac{\tilde{f}''(x)}{6}| \leq \frac{\varepsilon^2}{3} \cdot o(1) = 0 \text{ as } \varepsilon \to 0$$

$$\therefore \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\mathcal{L}_\varepsilon f(x) = -\frac{f''(x)}{6}.$$

Functions in C^3 and C^4

Corollary (1)

If $f \in C^3[-1,1]$ *then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{L}_\varepsilon f(x) = -\frac{f''(x)}{6}.$$

Functions in C^3 and C^4

Proof.

Refer to our previous proof of Theorem (1). Expand the Taylor series of $\tilde{f}$ to its third derivative term, i.e.

$$\tilde{f}(y) = \tilde{f}(x) + \tilde{f}'(x)(y-x) + \frac{\tilde{f}''(x)}{2}(y-x)^2 + \frac{\tilde{f}'''(x)}{6}(y-x)^3 + o(\varepsilon^3)$$

Performing the same integration as before, we obtain

$$|\frac{1}{\varepsilon^2}\mathcal{L}_\varepsilon f(x) + \frac{\tilde{f}''(x)}{6}| \leq 0 \text{ as } \varepsilon \to 0$$

□

Due to the symmetry of $\tilde{f}$, our odd degree terms will cancel as desired.

Functions in C^3 and C^4

Corollary (2)

If $f \in C^4[-1,1]$ *then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathcal{L}_\varepsilon f(x) = -\frac{f''(x)}{6}.$$

Convergence of Discrete Operator

To follow the results of Giné and Koltchinskii, we want to prove that $\mathcal{L}_{n,\varepsilon} f(x)$ and $\mathcal{L}_{\varepsilon} f(x)$ converge as the number of points n goes to infinity.

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It seems like on the interval with evenly spaced points, $|\mathcal{L}_{n,\varepsilon} f(x) - \mathcal{L}_{\varepsilon} f(x)| \to 0$ as $n \to \infty$. What does this convergence depend on? Placement of points, definition of f, something else?

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Convergence depends on how well points approximate the space and the relationship between metric and measure.

Measure-Theoretic Proof Sketch

Let (X, μ) be a measurable space such that for any natural number n, there exists a partition $\mathcal{A}_n$ of X into measurable cells $A_{n,i}$, with $\mu(A_{n,i}) = \sigma_n$ for some σ_n independent of i.

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$$\mathcal{T}_m f(x) = \frac{1}{\sigma_m} \int_{A_{m(x)}} (f(x) - f(y)) d\mu(y)$$

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$$\mathcal{L}_{n,m} f(x) = \frac{\sigma_n}{\sigma_m} \sum_{x_{n,j} \in A_m(x)} f(x) - f(x_{n,j})$$

Measure-Theoretic Proof Sketch

$$
\begin{aligned}
|\mathcal{L}_{n,m}f(x) - \mathcal{T}_m f(x)| &= \left| \frac{1}{\sigma_m} \int_{A_{m(x)}} f(y)d\mu(y) - \frac{\sigma_n}{\sigma_m} \sum_{x_{n,j} \in A_m(x)} f(x_{n,j}) \right| \\
&= \frac{\sigma_n}{\sigma_m} \left| \sum_{A_{n,j} \subset A_{m(x)}} \frac{1}{\sigma_n} \int_{A_{n,j}} f(x_{n,j}) - f(y)d\mu(y) \right| \\
&\leq \frac{\sigma_n}{\sigma_m} \sum_{A_{n,j} \subset A_{m(x)}} \frac{1}{\sigma_n} \left| \int_{A_{n,j}} f(x_{n,j}) - f(y)d\mu(y) \right| \\
&\leq \sup_{A_{n,j} \subset A_{m(x)}} \sup_{z \in A_{n,j}} |f(x_{n,j}) - f(z)| \leq \omega_f(A_n)
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\end{aligned}
$$

So the difference between $\mathcal{L}_{n,m}f(x)$ and $\mathcal{T}_m f(x)$ **at a given minimum level** n is bounded by **the worst oscillation over a single level-**n **cell.**

Connecting to a Metric

Now, suppose (X, μ) is equipped with a metric such that $\text{diam}(A_n) \to 0$ as $n \to \infty$, and that f is uniformly continuous.

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Since the difference between $\mathcal{L}_{n,m}f(x)$ and $\mathcal{T}_m f(x)$ is bounded by $\omega_f(A_n)$, as the diameter of A_n goes to 0, $|\mathcal{L}_{n,m}f(x) - \mathcal{T}_m f(x)| \to 0$.

Example: Convergence on Sierpinski Gasket

Consider a level-n approximation of the SG embedded in $\mathbb{R}^2$ with a measure that assigns cells of equal level the same volume.

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So any neighborhood can be partitioned into level-n cells. Applying the first result,

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Since $\text{diam}(A_n) \to 0$ as $n \to \infty$, for f uniformly continuous, we can conclude $|\mathcal{L}_{n,m}f(x) - \mathcal{T}_m f(x)| \to 0$.