Laplacian Eigenmaps and Orthogonal Polynomials

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University of Connecticut, 2023 NSF REU in Mathematics

Graph Laplacian Operators

We define a graph $G := \{V, E\}$ on $[-1, 1]$ where V is n equally spaced points $\{x_0 = -1, ..., x_{n-1} = 1\}$ and $(x_i, x_j) \in E$ if $|x_i - x_j| = \delta = \frac{2}{n-1}$ where δ represents the spacing between any two adjacent points.
Let W_G be the adjacency matrix and D_G be the degree matrix of G, where

$$W_{G_{i,j}} = \begin{cases} 1, \text{ if } |x_i - x_j| = \delta, i \neq j \\ 0, \text{ else} \end{cases} \qquad D_{G_{i,j}} = \begin{cases} \sum_{m=0}^{n-1} W_{G_{i,m}}, \text{ if } i = j \\ 0, \text{ else} \end{cases}.$$

Then, the Regular Laplacian matrix is defined by

$$L_{\text{reg}} := D_G - W_G.$$

The Probabilistic Laplacian matrix is defined by

$$L_{\text{prob}} := D_G^{-1}(D_G - W_G).$$

We define graph $G' := \{V, E \cup \{x_0, x_{n-1}\}\}$ from n evenly spaced points on a circle. The Periodic Laplacian is defined by

$$L_{\text{per}} := D_{G'} - W_{G'}.$$

$$L_{\text{reg}} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}, L_{\text{prob}} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}, L_{\text{per}} = \begin{bmatrix} 2 & -1 & 0 & \cdots & -1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

Defining Eigen-Coordinates

Define the eigen-coordinates as a map $\Phi_{v,w} : \{x_i\}_0^{n-1} \to \mathbb{R}^2$ such that

$$\Phi_{v,w}(x_i) := (v(x_i), w(x_i)) \in \mathbb{R}^2.$$

Let (v, w) be a set containing all the coordinates formed by two eigenfunctions

$$\{(v(x), w(x)) \mid x \in \{x_i\}_0^{n-1}\}.$$

Suppose L is a graph Laplacian of any kind. Sort its eigenvalues such that $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$. Our objective is to investigate the eigen-coordinates (v_1, v_ℓ).

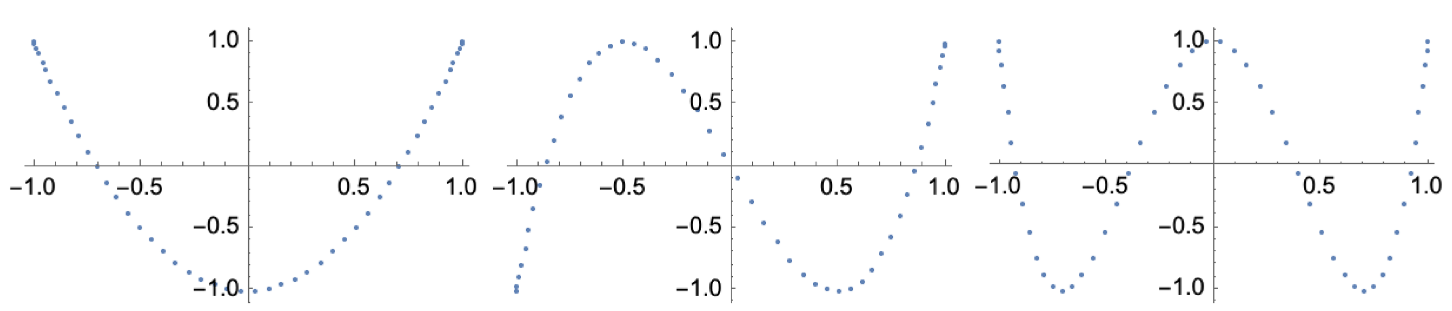


Fig. 1: From left to right, the images are eigenmaps of (v_1, v_2), (v_1, v_3), and (v_1, v_4) respectively.

Finding Eigenvalues and Eigenfunctions

Lemma. *Let L be a graph Laplacian matrix. If the middle $n-2$ rows are of the form*

$$r \begin{bmatrix} -1 & 2 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \ldots & \ldots & 0 & -1 & 2 & -1 \end{bmatrix}, r \in \mathbb{R}$$

then $Ae^{i\pi\alpha x} + Be^{-i\pi\alpha x}$ is an eigenvector, its corresponding eigenvalue is $2r(1 - \cos(\pi\alpha\delta))$.

Theorem. *Eigenvalues of L_{reg} are of the form $\lambda = 2(1 - \cos(\pi\alpha\delta))$.*

Theorem. *Eigenvectors of L_{reg} are of the form $f(x) = \cos(\pi\alpha x)$ (with c even) or $\sin(\pi\alpha x)$ (with c odd) where $\alpha = \frac{c}{\delta+2}$ and $c \in \mathbb{Z}$.*

Proof. Considering the endpoints $j = 0$ and $j = n-1$, we have:

$$Lf(-1) = Ae^{i\pi\alpha(-1)} + Be^{-i\pi\alpha(-1)} - (Ae^{i\pi\alpha(-1+\delta)} + Be^{-i\pi\alpha(-1+\delta)})$$
$$Lf(1) = Ae^{i\pi\alpha(1)} + Be^{-i\pi\alpha(1)} - (Ae^{i\pi\alpha(1-\delta)} + Be^{-i\pi\alpha(1-\delta)})$$

We need $Lf(-1) = \lambda f(-1)$ and $Lf(1) = \lambda f(1)$. By setting up the system of equations:

$$e^{-i\pi\alpha}(e^{-i\pi\alpha\delta} - 1) \begin{bmatrix} 1 & -e^{i\pi\alpha(2+\delta)} \\ -e^{i\pi\alpha(2+\delta)} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\begin{bmatrix} A \\ B \end{bmatrix}$ to be non-trivial (i.e., not the zero vector), the matrix $\mathbf{M} = \begin{bmatrix} 1 & -e^{i\pi\alpha(2+\delta)} \\ -e^{i\pi\alpha(2+\delta)} & 1 \end{bmatrix}$ must be singular. Thus, we can find α when $\det(\mathbf{M}) = \mathbf{0}$, leading to $\alpha = \frac{c}{\delta+2}$, where $c \in \mathbb{Z}$. Upon plugging α into $\mathbf{M}$, we find that $A = (-1)^c B$. Normalizing the solution, we obtain $f(x) = \cos(\pi\alpha x)$ if c is even and $\sin(\pi\alpha x)$ if c is odd. □

Theorem. *Eigenvalues of L_{prob} are of the form $\lambda = 1 - \cos(\pi\alpha\delta)$ and the eigenfunctions are of the form $\cos(\pi\alpha x)$ (with c even) or $\sin(\pi\alpha x)$ (with c odd) where $\alpha = \frac{c}{2}$, $c \in \mathbb{Z}$.*

Theorem. *Eigenvalues of L_{per} are of the form $\lambda = 2(1 - \cos(\pi\alpha\delta))$ and the eigenfunctions are of the form $e^{i\pi\alpha x}$ where $\alpha \in \{0, 1, \ldots, n-1\}$. When n is even, the eigenvalues are exactly $\lambda(0), \lambda(1), \ldots, \lambda(\frac{n}{2})$. When n is odd, the eigenvalues are exactly $\lambda(0), \lambda(1), \ldots, \lambda(\frac{n-1}{2})$.*

Correlation to Chebyshev Polynomials

Definition. *The Chebyshev Polynomials of the first kind are defined by $T_\ell(\cos(\theta)) = \cos(\ell\theta)$.*

We want to investigate the error, $E(x)$, between our eigen-coordinates (v_1, v_ℓ) and the Chebyshev polynomial T_ℓ:

$$E(x) = v_\ell(x) - T_\ell(v_1(x)).$$

Theorem. *The Chebyshev polynomials of $\sin(y)$*

$$T_\ell(\sin(y)) = \frac{i^{-\ell}}{2}(e^{i\ell y} + (-1)^\ell e^{-i\ell y}).$$

We illustrate with the following examples.

$$T_0(\sin(y)) = 1, \quad T_1(\sin(y)) = \sin(y), \quad T_2(\sin(y)) = -\cos(2y)$$
$$T_3(\sin(y)) = -\sin(3y), \quad T_4(\sin(y)) = \cos(4y), \quad T_5(\sin(y)) = \sin(5y).$$

Corollary. *$E(x) = v_\ell(x) - T_\ell(v_1(x)) = 0$ for L_{reg} and L_{prob}. This implies the eigen-coordinates (v_1, v_ℓ) for L_{reg} and L_{prob} are exactly Chebyshev polynomials.*

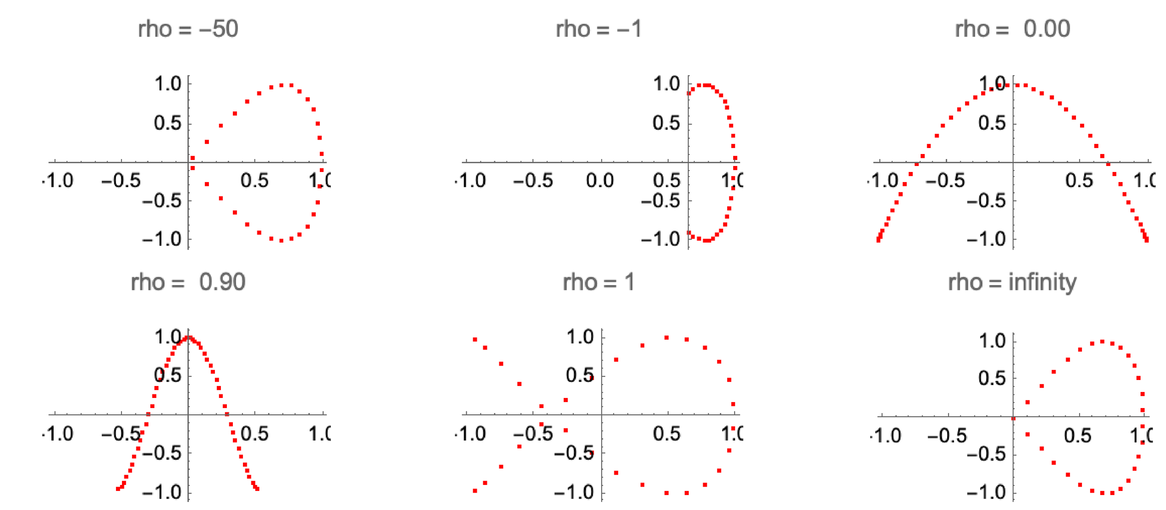
Robin Problems on $[-1, 1]$

Definition. *Suppose u is an eigenfunction of the Laplacian $L_{n\to\infty,k=1}$ with corresponding eigenvalue λ. The Robin Boundary Conditions are*

$$\begin{cases} u'' = -\lambda u & \text{on } (-1, 1) \\ \partial_n u = \rho u & \text{at } x = \pm 1 \end{cases}$$

where ∂_n is the outward normal derivative at $x = \pm 1$.

Solving the general Robin Boundary Condition, we have



Definition. *The Robin Boundary Conditions for the Probabilistic Laplacian are*

$$\begin{cases} L_{\text{prob}} u = -\lambda u & \text{at } x \in (-1, 1) \\ L_{\text{prob}} u = \sigma u & \text{at } x = \pm 1 \end{cases}.$$

Solving the Robin boundary conditions for L_{prob}, we have $\lim_{n\to\infty} \sigma = 0$, which implies that the eigen-coordinates for the L_{prob} are the Chebyshev polynomials of the first kind, $T_n(x)$.

Corollary. *The eigen-coordinates for the L_{reg} are the Chebyshev polynomials of the first kind.*

We also show the convergence of the Discrete Robin Problem for L_{reg} to the Continuous Robin Problem.

Further Directions

- Approximation of eigen-coordinates from random points on $[-1, 1]$, the half sphere $\mathbb{S}^2$, the Sierpinski Gasket, and other distributions.

Acknowledgements

This poster is based on joint work with Bobita Atkins, Ashka Dalal, Natalie Dinin, Tess McGuinness, Tonya Particks, and Genevieve Romanelli under the supervision of Bernard Akwei and Rachel Bailey. We thank Professors Luke Rogers and Alexander Teplyaev for their guidance throughout the REU. This work was made possible by the NSF grant DMS-1950543.

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