Resistance Scaling on Fractal Carpets

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Introduction

Definition: A contraction \( f \) on a metric space \((X,d)\) is a map \( f : X \to X \) such that \( \forall y \in [0,1], \forall a,b \in X, \) where \( d(f(a),f(b)) \leq d(a,b) \).

Definition: An iterated function system (IFS) is a finite set of contractions \( \{ f_i \} \) on a complete metric space. It is a fact that the Hutchinson operator on non-empty compact sets \( A \), defined as \( f(A) = \bigcup_i f_i(A) \), has a fixed point \( K \) such that \( f(K) = K \). Frequently \( K \) is called the fractal of the IFS. An example of such a \( K \) is the Sierpinski carpet. One can approximate \( K \) by composing \( f \) with itself repeatedly. Set \( F_0 = f \circ f \circ \cdots \circ f \). Let \( F_2 = [0,1]^2 \) and \((X,d)\) be \( R^2 \) with the Euclidean metric. Define \( F_n = f^n(\mathcal{F}) \).

Definition: The (total) energy of a ‘nice’ function \( f(x) \) on domain \( \Omega \) with ‘nice’ boundary is:

\[
E_0(f) = \int_{\Omega} |\nabla f|^2
\]

On \( F_n \), we set boundary conditions: \( f(0,x) = 0 \), \( f(1,x) = 1 \), and \( y_0 = 0 \) for \( y \in \{0,1\} \) and require continuity and finite energy on the interior. There is a unique harmonic function \( u_n \) that satisfies the conditions and minimizes \( E_0 \).

Definition: The effective resistance of \( F_n \) is defined as \( R_n = (\frac{1}{E_0(u_n)})^{-1} \).

Intuitively, we expect the resistance to go up as \( n \to \infty \). Imagine the holes in \( K \) as obstacles in a river. One can approximate \( K \) with itself repeatedly. Set \( F_0 = f \circ f \circ \cdots \circ f \). Let \( F_2 = [0,1]^2 \) and \((X,d)\) be \( R^2 \) with the Euclidean metric. Define \( F_n = f^n(\mathcal{F}) \).

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Theorem: (Barlow and Bass, 1990) There exists a constant \( \rho \geq 1 \) such that:

\[
\frac{1}{\rho^n} \leq R_n \leq 4\rho^n \quad n \geq 0
\]

\( \rho \) is called the resistance scaling factor, since it implies that \( R_n \approx \rho R_{n-1} \). The result proved by using graph approximations to build a ‘quilt’ function that is comparable to the energy-minimizing function - this was highly dependent on the rotational and reflective symmetry of the fractal.

Affine Carpet Resistances

The contractions for the SC are identical but with different fixed points. For the \( k \)-affine carpet we contract the four corner squares by \( k \), and the four remaining contractions map squares to rectangles, as in the \( 1 \)-affine carpet:

```
-0 1 2
-0 1 2
-0 1 2
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The key difficulty is that Barlow and Bass made heavy use of the rotational symmetry of squares and uniform edge weights in graph approximations. In affine carpets there are cells of different scales and rectangular eccentricities, rectangles are not diagonally symmetric, and graph approximations have heterogeneous weights. We have numerically investigated the edge weights for the graph approximations and examined several approaches to resolving the difficulties, but have not yet solved the problem. There is one numerical observation:

Conjecture: Let \( R^a \) and \( R^b \) be the effective resistances of the affine carpet mapped to \( 1 \times k_1 k_2 \) rectangles where \( k_i \in (0,\infty) \). Then:

\[
\frac{R^a_{k_1} R^{-a}_{k_2}}{R^{-a}_{k_2}} = \frac{R^{-a}_{k_1} R^a_{k_2}}{R^a_{k_2}}
\]

See Numerical Results for details.

Weighted Carpet Resistances

Consider the SC, instead of no center square, we weight differently from the other eight squares. Let \( t \) be the (non-negative) weight of the central cell, and \( s \) the weights of the other 8 cells, and set \( 8s + t = 1 \) so that the total mass is 1. Note that \( R_n \) in this case is a grid imposed on \([0,1]^2\) with cells of size \((1/3)^n\). Let \( C \) be a cell in \( F_n \), then \( C \) is the image of a sequence of contractions; let \( g \) be the central contraction, and suppose \( g \) was applied \( k \) times to \([0,1]^2\) to reach \( C \). We define \( u(t) = t^k \rho^n \). Here are approximate images of \( u_n \) and \( u_4 \) for \( t = 0.001 \):

We believe we can prove the following, but have not completed our write-up:

Conjecture: For any \( t \in [0,1/9] \):

\[
\frac{32}{3^2} \rho^n \leq R_n \leq 32 \rho^n
\]

Computation Technique

Averaging property: Let \( G = (V,E) \) be a connected graph with a symmetric weighting function \( g : E \to [0,\infty] \). Let \( I \subseteq V \) be the interior of \( G \), and let \( g : (V \setminus I) \to \mathbb{R} \) be boundary data. We call the unique function \( f \) such that \( f|_{V\setminus I} \equiv g \) and:

\[
f(x) = \left( \sum_{(y,x) \in E} g(y,x) f(y) \right) / \left( \sum_{y \in V} g(y,x) \right)
\]

for all \( x \in I \) a graph harmonic function.

Method of Relaxations: To approximate \( u_n \) on \( F_n \), we consider \( G_n \) or \( D_n \) with reciprocals of distances as edge weights. An arbitrary initial function is defined that matches the boundary data. Then, every interior vertex’s function value was replaced with the weighted average of its neighbor’s function values. This is called a relaxation. As the number of relaxations approaches infinity, the function on the shape approaches the true harmonic function.

Since within one affine carpet, there are rectangular cells of varying ‘eccentricities’, we had to analyze how \( \rho \) appears to stay constant, regardless of stretch:

![Resistance Scaling on Fractal Carpets](Image)

![Weighted Carpet Resistances](Image)

![Numerical Results](Image)

This is consistent with the hypothesized relationship, but more data is still needed.

Next steps:

- Compute deeper-level data for affine carpets and weighted carpets for better numerical evidence.
- Prove conjectures related to weighted carpets.
- Find a way to get around the symmetry issue for affine carpets.

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