Resistance Scaling on Fractal Carpets

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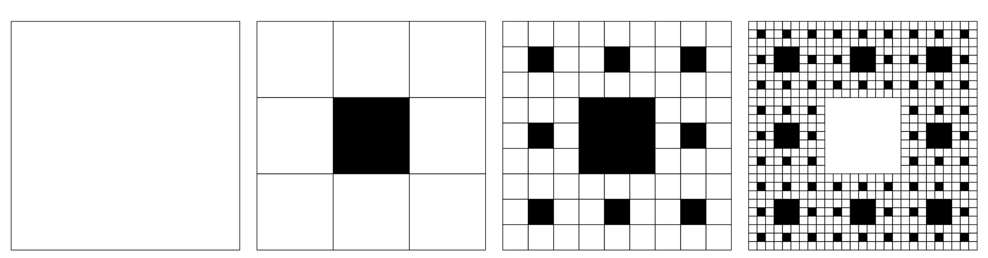
Introduction

Definition: A **contraction** f on a metric space (X,d) is a map $f : X \to X$ such that $\exists q \in [0,1)$, $\forall a,b \in X$, where $d(f(a), f(b)) \le q \cdot d(a,b)$.

Definition: An **iterated function system** (IFS) is a finite set of contractions $\{f_i\}$ on a complete metric space. It is a fact that the Hutchinson operator on non-empty compact sets A, defined as $f(A) = \cup_i f_i(A)$, has a fixed point K such that $f(K) = K$. Frequently K is called the fractal of the IFS. An example of such a K is the Sierpinski carpet. One can approximate K by composing f with itself repeatedly. Set $f^n := f \circ f \circ \cdots \circ f$. Let $I^2 = [0,1]^2$ and (X,d) be R^2 with the Euclidean metric. Define $F_n = f^n(I^2)$. Visually:



$level = 0$ $level = 1$ $level = 2$ $level = 3$

Definition: The (total) *energy* of a 'nice' function $f(x)$ on domain Ω with 'nice' boundary is:

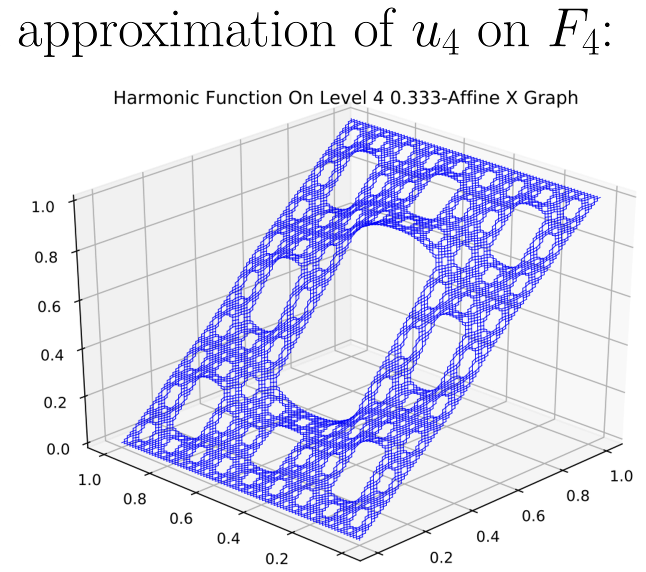
$$\mathcal{E}_\Omega(f) = \int_\Omega |\nabla f|^2$$

On F_n, we set boundary conditions: $f(0,x) = 0$, $f(1,x) = 1$, and $\frac{\partial f}{\partial n} = 0$ for $y \in \{0,1\}$ and require differentiability and finite energy on the interior. There is a unique *harmonic* function u_n that satisfies the conditions and minimizes $\mathcal{E}_{F_n}$.

Definition: The *effective resistance* of F_n is defined as $R_n = (\mathcal{E}_{F_n}(u_n))^{-1}$.

Intuitively, we expect the resistance to go up as $n \to \infty$. Imagine the holes in F_n as obstacles in a river. The more obstacles, the higher the resistance is; it is more difficult for water to travel through the region.

Here is an approximation of u_4 on F_4:



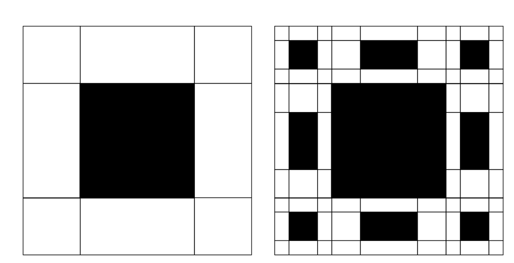
Theorem: (Barlow and Bass, 1990) There exists a constant $\rho \ge 1$ such that:

$$\frac{1}{4}\rho^n \le R_n \le 4\rho^n \quad n \ge 0$$

ρ is called the *resistance scaling factor*, since it implies that $R_n \approx \rho R_{n-1}$. The result proved by using graph approximations to build a 'quilt' function that is comparable to the energy-minimizing function - this was highly dependent on the rotational and reflective symmetry of the fractal.

Affine Carpet Resistances

The contractions for the SC are identical but with different fixed points. For the ***k*-affine carpet** we contract the four corner squares by k, and the four remaining contractions map squares to rectangles, as in this $\frac{1}{4}$-affine carpet:



$level = 1$ $level = 2$

The key difficulty is that Barlow and Bass made heavy use of the rotational symmetry of squares and uniform edge weights in graph approximations. In affine carpets there are cells of different scales and rectangular eccentricities, rectangles are not diagonally symmetric, and graph approximations have heterogeneous weights. We have numerically investigated the edge weights for the graph approximations and examined several approaches to resolving the difficulties, but have not yet solved the problem. There is one numerical observation:

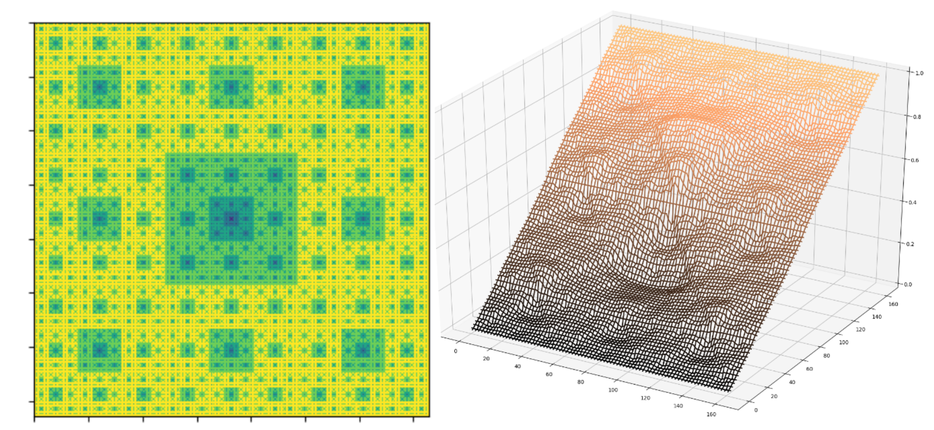
Conjecture: Let R_n^1 and R_n^2 be the effective resistances of the affine carpet mapped to $1 \times k_1, k_2$ rectangles where $k_i \in (0,\infty)$. Then:

$$\frac{R_{n+1}^1}{R_n^1} = \frac{R_{n+1}^2}{R_n^2}$$

See Numerical Results for details.

Weighted Carpet Resistances

Consider the SC, instead of no center square, we weigh it differently from the other eight squares. Let t be the (non-negative) weight of the central cell, and s the weights of the other 8 cells, and set $8s + t = 1$ so that the total mass is 1. Note that F_n in this case is a grid imposed on $[0,1]^2$ with cells of size $(1/3^n)^2$. Let C be a cell in F_n, then C is the image of a sequence of contractions; let g be the central contraction, and suppose g was applied k times to $[0,1]^2$ to reach C. We define $\mu_n(C) = t^k s^{n-k}$. Here are approximate images of μ_n and u_4 for $t = 0.001$:



We believe we can prove the following, but have not completed our write-up:

Conjecture: For any $t \in [0, 1/9]$.

$$\frac{1}{32}\rho^n \le R_n \le 32\rho^n$$

Computation Technique

Averaging property: Let $G = (V,E)$ be a connected graph with a symmetric weight function $g : E \to [0,\infty)$. Let $I \subset V$ be the *interior* of G, and let $g : (V \setminus I) \to \mathbb{R}$ be *boundary data*. We call the unique function f such that $f|_{V\setminus I} \equiv g$ and:

$$f(x) = \left(\sum_{(y,x)\in E} g(y,x) f(y) \right) \Big/ \left(\sum_{(y,x)\in E} g(y,x) \right)$$

for all $x \in I$ a **graph harmonic** function.

Method of Relaxations: To approximate u_n on F_n, we consider G_n or D_n with reciprocals of distances as edge weights. An arbitrary initial function is defined that matches the boundary data. Then, every interior vertex's function value was replaced with the weighted average of its neighbor's function values. This is called a **relaxation**. As the number of relaxations approaches infinity, the function on the shape approaches the true harmonic function.

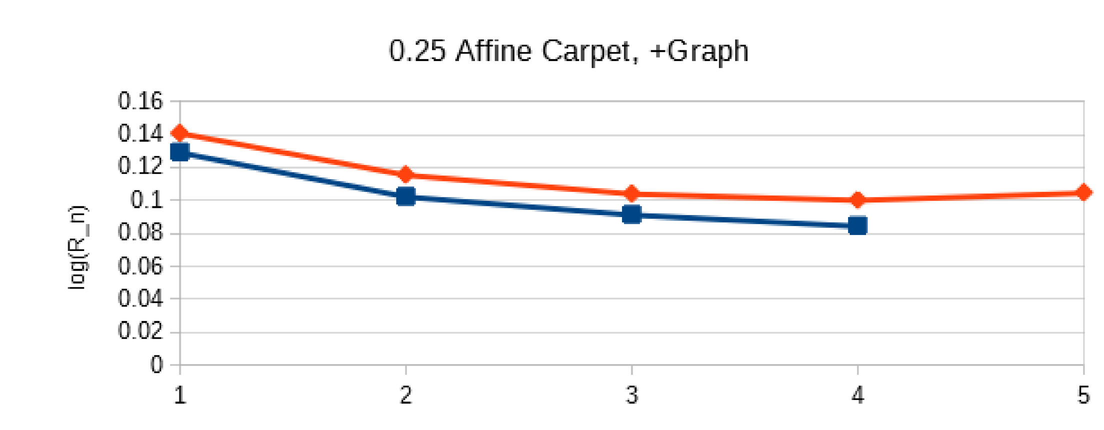
Numerical Results

Since within one affine carpet, there are rectangular cells of varying 'eccentricities', we had to analyze how $\rho \approx \frac{R_n}{R_{n-1}}$ changes with the amount of stretching the carpet experienced. The following graphs show how ρ appears to stay constant, regardless of stretch:

	0.25 Affine Cross Carpet						
Level	1x0.125	1x0.25	1x0.5	1x1	1x2	1x4	1x8
1	2.495	2.495	2.495	2.495	2.495	2.495	2.495
2	2.838	2.838	2.838	2.838	2.838	2.839	2.839
3	3.143	3.143	3.144	3.143	3.143	3.143	3.143
4	3.444	3.444	3.445	3.443	3.444	3.444	3.444
5	3.760	3.759	3.760	3.746	3.742	3.831	3.830

	0.25 Affine $\times$ Carpet						
Level	1x0.125	1x0.25	1x0.5	1x1	1x2	1x4	1x8
1	1.157	1.157	1.157	1.157	1.157	1.157	1.157
2	1.332	1.332	1.332	1.332	1.332	1.332	1.332
3	1.495	1.495	1.495	1.494	1.495	1.495	1.495
4	1.662	1.662	1.662	1.658	1.662	1.662	1.662
5	1.841	1.840	1.841	1.832	1.841	1.841	1.841
6	2.032	2.033	2.034	2.034	2.034	2.034	2.034

Recall that if $\rho^n \approx R_n$, then $n\log(\rho) \approx \log(R_n)$ gives us a way to check if linear behavior numerically supports that a ρ exists; here is an $(n, \log(R_n))$ plot:



This is consistent with the hypothesized relationship, but more data is still needed.

Next steps:

- Compute deeper-level data for affine carpets and weighted carpets for better numerical evidence,
- Prove conjectures related to weighted carpets,
- Find a way to get around the symmetry issue for affine carpets.

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[1] Barlow, M. T., and R. F. Bass. "On the Resistance of the Sierpinski Carpet." *Proceedings: Mathematical and Physical Sciences*, vol. 431, no. 1882, 8 Nov. 1990.

[2] Barlow, M T, et al. "Resistance and Spectral Dimension of Sierpinski Carpets." *Journal of Physics A: Mathematical and General*, vol. 23, no. 6, 1990, pp. L253–L258.

[3] Doyle, Peter G., and J. Laurie Snell. "Random Walks and Electric Networks." 2009.