Decimation Structure of the Spectra of Self-Similar Groups

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Outline

1. Self-Similar groups
2. The Hanoi Towers Group
3. Spectral Similarity
4. The Grigorchuk Method
5. Future Research

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4. The Grigorchuk Method
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Definition (Spectrum [Str06])

The spectrum of a finite dimensional operator is the set of eigenvalues of that operator, denoted $\sigma(x)$.

Definition (Restriction of an automorphism [KSW12])

Let X^* be an infinite tree over an alphabet X^* Let g be a tree automorphism of X^*, and let $x \in \mathsf{X}$. Then $x\mathsf{X}^*$ is naturally isomorphic to X^*. Since g is an automorphism, restricting g to $x\mathsf{X}^*$ gives an automorphism on $x\mathsf{X}^*$. In turn, this gives another automorphism on X^*, which we denote $g|_x$.

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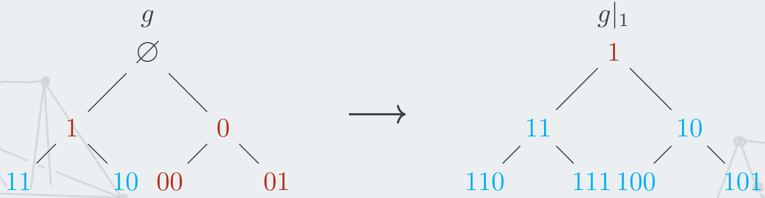

Self Similar Groups

Definition (Self similar groups [KSW12; Nek05])

Let G be a group that acts faithfully on X^*. Then G is a self similar group if, for all $g \in G$ and $x \in \mathsf{X}$ there is a $h \in G$ and $y \in \mathsf{X}$ such that $g(xw) = yh(w)$. Equivalently, for all $g \in G$ and $x \in \mathsf{X}$, we have $g|_x \in G$.

Example: The Hanoi Towers Group

The Hanoi Towers game is played on k pegs with n disks. The disks are labeled 1 through n. Players can move disks from one adjacent peg to another, but cannot place a larger disk on top of a smaller disk. [GS08]

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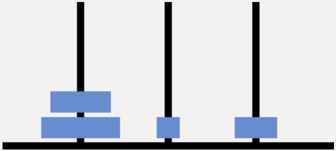
Let $\mathsf{X} = \{0, 1, \ldots, k-1\}$. Then we can uniquely represent each game state as a word in X^n.



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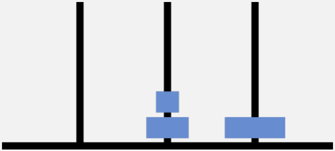

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Definition (The Hanoi Towers group [GS08])

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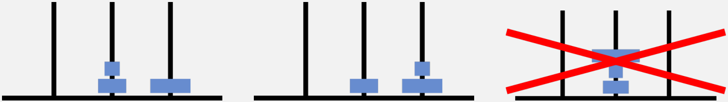
The Hanoi Towers group is defined in terms of its action on the set of game states of the Hanoi Towers game. Note that this action is always unique due to the rules of the game



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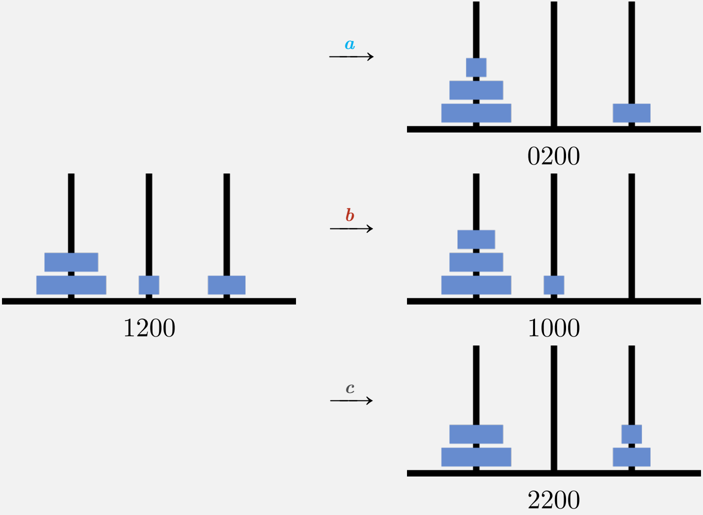
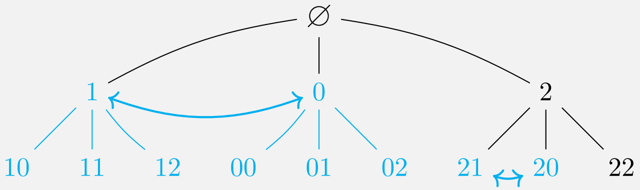


Example: The Hanoi Towers Group

Since we can represent each game state as a word over $\mathsf{X} = \{0, 1, 2\}$, H also acts on X^*.



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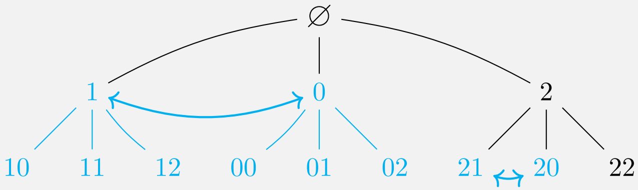


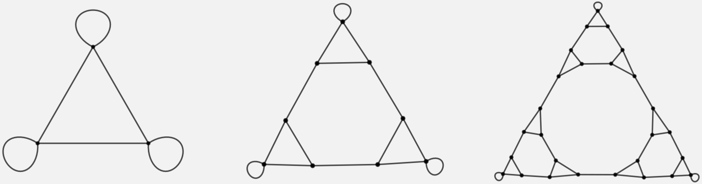
We can specify a group element by its permutation on the first level and what it restricts to on lower levels. For Hanoi:

$$\boldsymbol{a} = (0\ 1)(\mathrm{id}, \mathrm{id}, \boldsymbol{a})$$
$$\boldsymbol{b} = (0\ 2)(\mathrm{id}, \boldsymbol{b}, \mathrm{id})$$
$$\boldsymbol{c} = (1\ 2)(\boldsymbol{c}, \mathrm{id}, \mathrm{id})$$

Schreier graphs

Definition (Schreier graphs [GS08])

Let G be a self similar group acting on X^*. Let S be a finite generating set of G. Then the nth Schreier graph of G with respect to S is a directed graph with vertices X^n and an edge from x to y if there is a $g \in S$ such that $gx = y$.



Spectral Similarity

Definition (Spectral similarity [MT03])

We call an operator H *spectrally similar* to an operator H_0 if we have complex functions φ_0 and φ_1 such that, when defined,

$$U^*(H-z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$$

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Spectral similarity is a way to relate the spectra of two different operators. If we take

$$R(z) = \frac{\varphi_1(z)}{\varphi_0(z)}$$

we see that $z \in \sigma(H)$ means $R(z) \in \sigma(H_0)$.
In our context, this means relating operators on graph approximations of fractals or Schreier graphs of self-similar groups.

Example of Spectral Similarity

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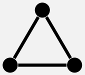

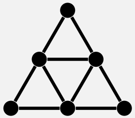

$\sigma(SG_1) = \{\frac{3}{2}, \frac{3}{2}, 0\}$

$\sigma(SG_2) = \{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 0\}$

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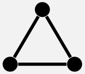

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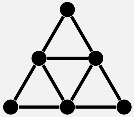

$\sigma(SG_2) = \{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 0\}$

Plugging the values from $\sigma(SG_2)$ into R gives you the values of $\sigma(SG_1)$. This gives a clear representation of a simple case of spectral similarity.

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We represent the generators with permutational matrices recursively by,

$$\boldsymbol{a}_0 = \boldsymbol{b}_0 = \boldsymbol{c}_0 = [1]$$

$$\boldsymbol{a}_{n+1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \boldsymbol{a}_n \end{bmatrix} \boldsymbol{b}_{n+1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{b}_n & 0 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{c}_{n+1} = \begin{bmatrix} \boldsymbol{c}_n & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

where each block is of size $3^n \times 3^n$ for $n \geqslant 0$. [GS08]

The Grigorchuk Method: The Adjacency Matrix

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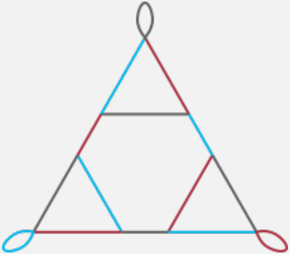
$$\Delta_n = \boldsymbol{a}_n + \boldsymbol{b}_n + \boldsymbol{c}_n$$

Equivalently,

$$\Delta_n = \begin{bmatrix} \boldsymbol{c}_n & 1 & 1 \\ 1 & \boldsymbol{b}_n & 1 \\ 1 & 1 & \boldsymbol{a}_n \end{bmatrix}$$

The Grigorchuk Method: The Adjacency Matrix

The adjacency for level 2 of the Hanoi Towers group;



$$\Delta_2 = \left[\begin{array}{ccc|ccc|ccc} c & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & a \end{array}\right]$$

Finding the spectrum

We now introduce a new variable x, [GNS15]. Define $\Delta_n(x) = \Delta_n - xI$, and the spectrum of Δ_n is all x where $|\Delta_n(x)| = 0$.
For example, level 2, we have

$$\left[\begin{array}{ccc|ccc|ccc}
\boldsymbol{c}-x & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -x & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & -x & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & -x & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & \boldsymbol{b}-x & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & -x & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & -x & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & -x & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \boldsymbol{a}-x
\end{array}\right]$$

Moving to a two-dimensional dynamic system

Our goal is to write the determinant of $\Delta_n(x)$ in terms of the determinant of $\Delta_{n-1}(x)$. Unfortunately, the linear algebra does not allow us to do this (at least, not yet). [GS08]

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Instead, we take $\Delta_n(x,y) = \Delta_n(x) - (1-y)d_n$, where

$$d_{n+1} = \begin{bmatrix} 0 & 1_n & 1_n \\ 1_n & 0 & 1_n \\ 1_n & 1_n & 0 \end{bmatrix}$$

For example:

$$\Delta_1(x,y) = \begin{bmatrix} 1-x & y & y \\ y & 1-x & y \\ y & y & 1-x \end{bmatrix}$$

Auxiliary spectrum

Now we can express the determinant of $\Delta_n(x, y)$ in a recursive formula

$$|\Delta_n(x, y)| = P_n(x, y)|\Delta_{n-1}(F(x, y))|$$

where $F(x, y) = (x', y')$ is a rational function in x and y and $P_n(x, y)$ is a polynomial. [GS08]

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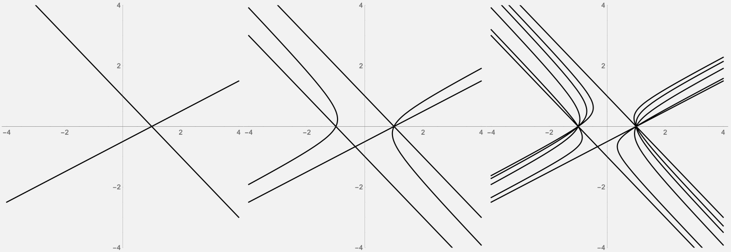
The zeroes of $\Delta_n(x, y)$ give us the auxiliary spectrum



Images of the auxiliary spectrum of levels 1, 2, and 3 [GS06].

Converting from the auxiliary spectrum to the Schreier spectrum

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Formally, we want to find Ψ and f that make this diagram commute.

As it turns out, Ψ and f do exist, and $f(x) = x^2 - x - 3$. [GS08] If $x \in \sigma(\Delta_n(x))$, then $f(x) \in \sigma(\Delta_{n-1}(x))$. We can use this fact to find the spectrum for any level.

Future research

- Extend a theorem of Malozemov and Teplyaev [MT03] about gluing spectrally similar objects to self similar groups.
- Extend a theorem of Nekrashevych and Teplyaev [NT08] about symmetries of an object and spectral similarity to self similar groups.
- Unify the above results to explain when a semi-conjugacy map for the Grigorchuk method exists and when it does not.

Thank you!

Rostislav Grigorchuk, Volodymyr Nekrashevych, and Zoran Sunić. "From self-similar groups to self-similar sets and spectra". In: *Fractal geometry and stochastics V*. Vol. 70. Progr. Probab. Birkhäuser/Springer, Cham, 2015, pp. 175–207. DOI: 10.1007/978-3-319-18660-3_11. URL: https://doi.org/10.1007/978-3-319-18660-3_11.

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