Decimation Structure of Self-Similar Group Spectra

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Abstract

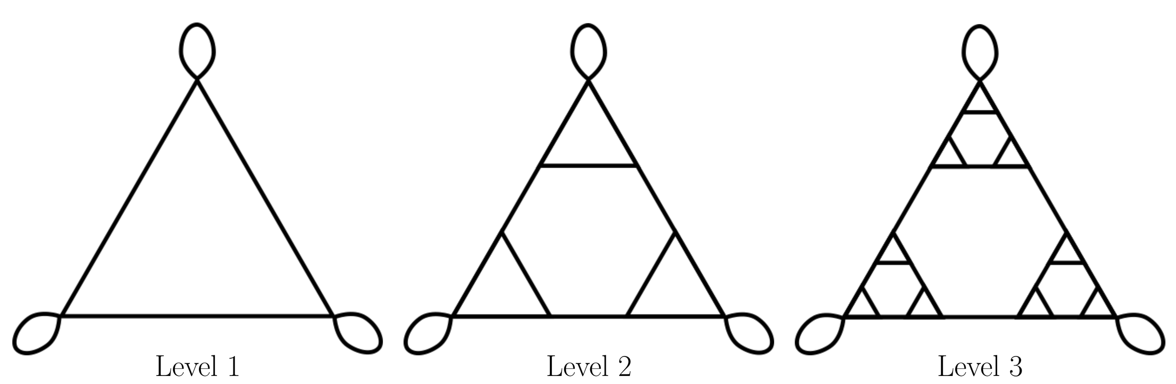
Physicists and mathematicians have used the self-similar nature of certain fractals to develop and study analytical structures on fractal spaces. We examine the analytical structure of a class of fractals that arise as limit sets of the Schreier graphs of the action of self-similar groups on infinite n-ary trees. In particular, we consider how the spectrum of a Laplacian operator on one level of a Schreier graph relates to the spectrum on the next level, a technique known as spectral decimation.

Grigorchuk and collaborators have developed a method to spectrally decimate Schreier graphs of several important self-similar groups, and have derived significant consequences about the structure of amenable groups. We discuss their method and propose an alternative method to admit spectral decimation to the Hanoi Towers through the Honeycomb method.

Definitions

Definition 1 (Self Similar Groups [4, 6]). Let X be a finite alphabet, and let X^* be the set of all finite words over X with a natural tree structure. Let G be a group that acts faithfully on X^*. Then G is a *self similar group* if, for all $g \in G$ and $x \in \mathsf{X}$ there is $h \in G$ and $y \in \mathsf{X}$ such that $g(xw) = yh(w)$.

Definition 2 (Schreier Graphs [3]). Let G be a self similar group acting on X^*. Let S be a finite generating set of G. Then the n-level *Schreier graph* of G with respect to S is a directed graph with vertices X^n and an edge from x to y if there is a $g \in S$ such that $gx = y$.



Hanoi Schreier graphs

Definition 3 (Adjacency Operator). The adjacency operator of a function $\varphi \in \ell^2(\Gamma_n)$ is defined by

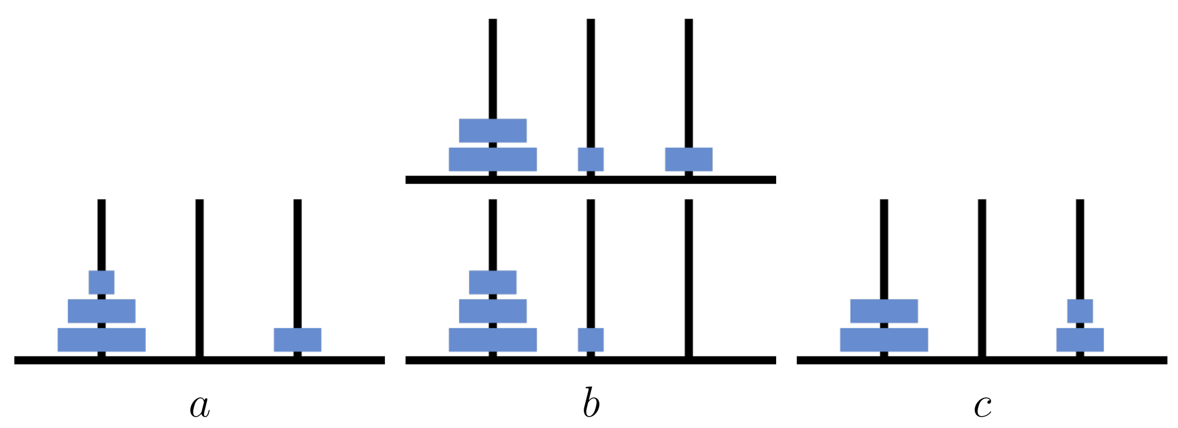
$$\Delta_n \varphi(x) = \sum_{s \in S} \varphi(sx)$$

for a vertex $x \in \mathsf{X}^n$.

Definition 4 (Spectral Decimation [1]). Let G be a self similar group, and let Δ_n be the adjacency matrix of the n-level Schreier graph of G. We say that G admits *spectral decimation* if we have a rational function R and finite set E such that $\sigma(\Delta_n) = \bigcup_{i=0}^{n} R^{-1}(E)$.

The Hanoi Towers Group

We consider as an example the Hanoi Towers group. The Hanoi Towers game is a game on three pegs that represents configurations with n disks as words of length n over $\mathsf{X} = \{0, 1, 2\}$. The Hanoi Towers group is described in terms of its action on game configurations. We have $H = \langle a, b, c \rangle$ where a represents making a move between the 0 and 1 pegs, b represents a move between 0 and 2, and c represents a move between 1 and 2.



Spectral Decimation of the Hanoi Towers Group

Spectral Decimation via a Two Dimensional Spectrum

We follow the method in [3] and [2]. For the Hanoi Towers group to admit spectral decimation, we need to be able to write the spectrum of the n-level Schreier graph in terms of the spectrum of the $(n-1)$-level Schreier graph. Equivalently, if we can find a recursive formula for the determinant of the adjacency matrices, we can find the zeroes in terms of previous levels. We can use the self-similar nature of the Hanoi Towers group to produce this recursive formula. We let

$$a_0 = b_0 = c_0 = [1],\ a_{n+1} = \begin{bmatrix} 0_n & I_n & 0_n \\ I_n & 0_n & 0_n \\ 0_n & 0_n & a_n \end{bmatrix},\ b_{n+1} = \begin{bmatrix} 0_n & 0_n & I_n \\ 0_n & b_n & 0_n \\ I_n & 0_n & 0_n \end{bmatrix},\ c_{n+1} = \begin{bmatrix} c_n & 0_n & 0_n \\ 0_n & 0_n & I_n \\ 0_n & I_n & 0_n \end{bmatrix}$$

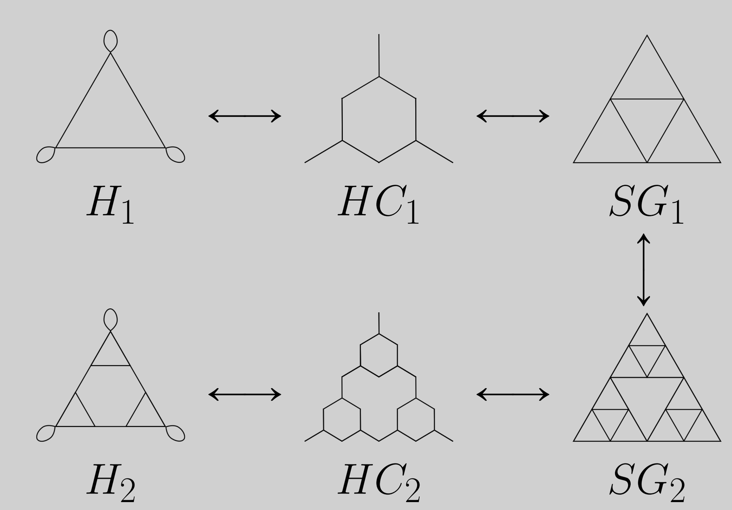
$$\Delta_n(x, y) = \Delta_n(x) + (y-1)\, d_n \text{ where } d_0 = [1] \text{ and } d_{n+1} = \begin{bmatrix} 0_n & I_n & I_n \\ I_n & 0_n & I_n \\ I_n & I_n & 0_n \end{bmatrix}.$$

We must move to a two-dimensional system because the one-dimensional system is not enough to express the determinant recursively. Hence, we take $P_n(x, y)$ and $F(x, y)$ as rational polynomials. For $n \geqslant 2$, we obtain the recursive formula $|\Delta_{n+1}(x, y)| = P_n(x, y)\, |\Delta_n(F(x, y))|$. Finally, we find a map that sends the two dimensional dynamic system to a one dimensional spectrum. Under that map, our two dimensional system is semi-conjugate to the one dimensional map $f(x) = x^2 - x - 3$, and the spectrum of Δ_n is

$$\{3\} \cup \bigcup_{i=0}^{n-1} f^{-i}(0) \cup \bigcup_{i=0}^{n-2} f^{-i}(-2).$$

Spectral Decimation via transformations to the Sierpinski Gasket

According to a lemma in [5], we can glue together spectrally similar objects to get other spectrally similar objects. The Sierpinski gasket admits spectral decimation, and the Delta-Wye transformation and 2-cell to 1-cell transformation also hold a spectral similarity. Therefore, we can move from the Hanoi Schreier graphs to the Sierpinski gasket via the transformations on the right and then spectrally decimate the Sierpinski gasket to obtain spectral decimation on the Hanoi graphs.



This process can only be done for the probability Laplacian, defined as

$$\Delta_n f(x) = f(x) - \frac{1}{\deg_n(x)} \sum_{(x,y) \in E(\Gamma_n)} f(y)$$

This gives us the spectral decimation map $R(x) = x(5 - 3x)$.

However, since the Hanoi Schreier graphs are all 3-regular (in general, Schreier graphs are k-regular for some k), the probability Laplacian can be related to the adjacency operator by $\Delta_{prob} = I - \frac{1}{3}\Delta_{adj}$.

Therefore, via the process on the right, we obtain the spectral decimation map $R(x) = x(5-3x)$ for the probability Laplacian. Conjugating through the map between the probability Laplacian and adjacency operator gives us the same map as above,

$$f(x) = x^2 - x - 3$$

Conclusion

The first method was pioneered by Grigorchuk in the 1980s, and has been successfully used to provide spectral decimation results for other self-similar groups. Unfortunately, this method has not been fully generalized or explained. We developed the second method, which is fully explained by [5]. However, it depends on the Schreier graph being spectrally similar to a graph with spectrally decimation. Our research has focused on uniting these two methods and understanding when and why they work.

Future Research

We are following three directions to provide a stronger theoretical foundation for the Grigorchuk approach.

1. Extend a theorem of Malozemov and Teplyaev [5] about gluing spectrally similar objects to self similar groups.
2. Extend a theorem of Nekrashevych and Teplyaev [7] about a group of symmetries of an object and spectral similarity to self similar groups.
3. Unify the above results to explain when a semi-conjugacy map for the Grigorchuk method exists and when it does not.

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