Hedging by Sequential Regression in Discrete Market Models

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Binomial Asset Pricing Model

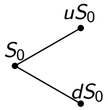
Two Components to Market

Stock (risky)

Random Variable (S_n) determined by sequence of "coin flips" ($\omega_1, ..., \omega_n$)

Money Market (risk-free)

Investment with constant return $M_{n+1} = (1 + r)M_n$ for some fixed interest rate $r > 0$



Example of a 1-period Binomial Model

Securities and Arbitrage

Definition

A **derivative security** is a security based on an underlying asset S_n which pays out some value V_n depending on the results of the coin toss.

Definition

Arbitrage is a trading strategy that begins with no money, has positive probability of profit, and zero probability of loss.

Fundamental Question of Arbitrage Pricing Theory

What price should a derivative security be sold at so there is a unique optimal hedging strategy that eliminates loss?

Options

European Call Option

$V_N = (S_N - K)^+$, where S_N is the stock price at time N and K is the agreed-upon strike price.

Lookback Option

$V_N = \max\limits_{0 \le i \le N} (S_i - S_N)$, where S_i is the stock price at time $i \le N$ and S_N is the stock price at time N.

Asian Call Option

$V_N = (\frac{\sum_{i=0}^{N} S_i}{N} - K)^+$, where S_N is the stock price at time N and K is the agreed-upon strike price.

Probability Theory and Random Variables

Definition

A **probability space** $(\Omega, \mathbb{P})$ consists of

- A set Ω, called a **probability space**
- A probability measure $\mathbb{P} : \mathcal{P}(\Omega) \to [0, 1]$ such that
  1. $\mathbb{P}(\Omega) = 1$
  2. If $A, B \subseteq \Omega, A \cap B = \varnothing$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Definition

Let $(\Omega, \mathbb{P})$ be a finite probability space. A **random variable** $X : \Omega \to A$ for some set A is a real-valued function on Ω.

Martingales

Definition

Let $M_0, M_1, ..., M_n$ be a **stochastic process**, or a sequence of random variables indexed by time, with each M_n depending on the first n coin tosses. This is an **adapted stochastic process.**

Example

The stock price S_n at time n is an adapted stochastic process.

Definition

A **martingale** is a stochastic process $M_0, M_1, ...$ such that $M_n = \mathbb{E}_n(M_{n+1})$.

Risk-Neutral Probabilities

Definition

Given a market $\{\Omega, \mathbb{P}, \{S_n\}_{n\geq 0}, r\}$, a **risk-neutral probability measure** is some $\tilde{\mathbb{P}} : \Omega \to [0,1]$ such that

- $\tilde{\mathbb{P}}(\omega) = 0 \iff \mathbb{P}(\omega) = 0$ for all $\omega \in \Omega$
- The discounted prices of all assets are martingales – i.e. $V_n = \frac{1}{(1+r)}\tilde{\mathbb{E}}_n(V_{n+1})$ for all securities V $(\implies V_0 = \frac{1}{(1+r)^N}\tilde{\mathbb{E}}(V_N)$ for all securities $V)$.

Example

The risk-neutral probabilities for a stock S_n on a binomial asset model are given by:
$\tilde{p} = \frac{1+r-d}{u-d}, \tilde{q} = \frac{u-1-r}{u-d}$

Hedging by Backward Recursion

Delta-Hedging Formula (Shreve)

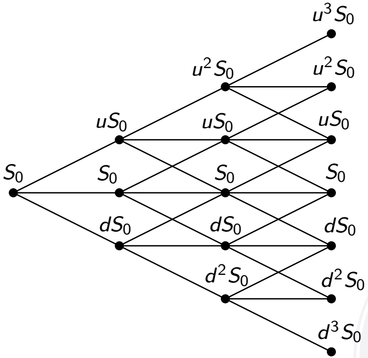
Let $V_n(\omega_1, \dots, \omega_n) = \frac{1}{(1+r)^{N-n}}\tilde{\mathbb{E}}[V_N \mid \omega_1, \dots, \omega_n]$. Then

$$\Delta_n(\omega_1, \dots, \omega_n) = \frac{V_{n+1}(\omega_1, \dots, \omega_n, H) - V_{n+1}(\omega_1, \dots, \omega_n, T)}{S_{n+1}(\omega_1, \dots, \omega_n, H) - S_{n+1}(\omega_1, \dots, \omega_n, T)}.$$

Wealth Process

$$X_{n+1} = (1+r)(X_n - \Delta_n S_n) + \Delta_n S_{n+1}$$

Trinomial Model



The trinomial model is an example of an incomplete market. It is slightly more accurate than the binomial model under appropriate circumstances, in particular for more complex options and over small time intervals.

Trinomial Model

Exercise

Show that a two-step trinomial model is incomplete.

Wealth Process Revisted

The two formulae below yield 12 equations for our 8 unknowns, making our solution over-determined.

$$X_1(\omega_1) = (1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1(\omega_1)$$

$$X_2(\omega_1\omega_2) = (1+r)(X_1(\omega_1) - \Delta_1 S_1(\omega_1)) + \Delta_1(\omega_1) S_2(\omega_1\omega_2)$$

Basic Problem

Problem

Given $V_N \in \mathcal{L}^2(P)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $c \in \mathbb{R}$, minimize

$$\mathbb{E}[(V_N - c - G_T(\xi))^2]$$

over all $\xi \in \Theta$, where Θ is the set of all predictable processes ξ such that $\xi_k \Delta S_k \in \mathcal{L}^2(P)$ and $G(\xi) := \sum_{j=1}^{k} \xi_j \Delta S_j$.

Framing the Problem

Interpretation

- S is the price process of some risky asset.
- V_N represents the random loss (contingent claim).
- c is the initial capital.
- $G_T(\xi)$ is the sum of all gains from trade up to time T.

With this interpretation, the goal in the basic problem is to minimize the expected net square loss experienced by the investor through choosing a trading strategy.

Decomposition of Processes

Semimartingale Decomposition

The process X can be written in its almost-surely unique **Semimartingale Decomposition** as

$$X = X_0 + M + A,$$

where $M \in \mathcal{L}^2$ is a martingale with $M_0 = 0$ and $A \in \mathcal{L}^2$ is a predictable process with $A_0 = 0$.

Föllmer-Schweizer Decomposition

For a contingent claim $V_N \in \mathcal{L}^2$, the **Föllmer-Schweizer Decomposition** of V_N is given by

$$V_N = V_0 + \sum_{j=0}^{N-1} \xi_j \Delta S_{j+1} + L_N$$

Formulae for ξ_n and L_N

For an $\mathcal{F}_n$ measurable event ω, we have the following formulae:

Recursive Formula for ξ

$$\xi_n(\omega) = \frac{\mathrm{Cov}\left(V_N - \sum_{j=n}^{N-1} \xi_j \Delta S_{j+1}, \Delta S_{n+1} \mid \omega\right)}{\mathrm{Var}(\Delta S_{n+1} \mid \omega)}$$

Formula for L

$$L_n(\omega) = \mathbb{E}[V_N - \sum_{j=0}^{N-1} \xi_j \Delta S_{j+1} | \omega] - \mathbb{E}[V_N - \sum_{j=0}^{N-1} \xi_j \Delta S_{j+1}]$$

The Result

Theorem

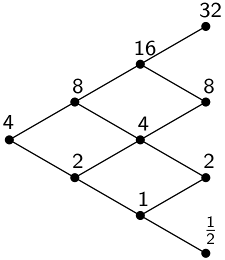
In the binomial model, Föllmer and Schweizer's (1989) method of hedging by sequential regression is equivalent the delta-hedging formula described in Shreve (2004). That is, given an $\mathcal{F}_n$ measurable event ω,

$$\xi_n(\omega) = \frac{Cov\left(V_N - \sum_{j=n}^{N-1} \xi_j \Delta S_{j+1}, \Delta S_{n+1} \mid \omega\right)}{Var(\Delta S_{n+1} \mid \omega)}$$

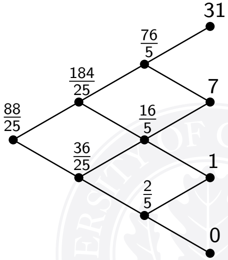
$$= \frac{V_{n+1}(\omega H) - V_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} = \Delta_n(\omega)$$

Equivalence Example

Find ξ_0/Δ_0 for the binomial market shown below:



Stock Prices



Discounted Option Values

Market Perturbations

Decomposition of ΔS

Consider the change in stock price at each time step, decomposed as

$$\Delta S_n = \lambda \Delta t + \sigma \Delta W$$

Market Perturbations

Define a **market perturbation**, for some $\varepsilon, \lambda', \sigma' \in \mathbb{R}$ as

$$\Delta S_n^{\varepsilon} = (\lambda + \varepsilon \lambda')\Delta t + (\sigma + \varepsilon \sigma')\Delta W$$

Stability of Sequential Regression

Question

We consider the stability of sequential regression under market perturbations, that is, as $\epsilon \to 0$, does $\xi^{\epsilon} \to 0$.

Theorem

The hedging process ξ is stable under market perturbations. I.e.

$$\lim_{\varepsilon \to 0} \xi_k^{\varepsilon}(\omega) = \xi_k^{0}(\omega)$$

for all $k \in \{0, \ldots, N-1\}$ and all $\mathcal{F}_k$-measurable ω.

Asymptotic Analysis

Theorem

In the finite, discrete time setting, the perturbed hedge ξ_n^ε converges asymptotically to ξ_n^0. That is,

$$\xi_n^\varepsilon \xrightarrow{\varepsilon\to 0} \xi_n^0$$
$$\frac{\xi_n^\varepsilon - \xi_n^0}{\varepsilon} \xrightarrow{\varepsilon\to 0} \tilde{\xi}_n,$$

where $\tilde{\xi}_n$ is a predictable process based on market parameters.

Thank You!