Hedging by Sequential Regression in Discrete Market Models

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Overview

The field of mathematical finance has long been concerned with the fair valuation of derivative securities. The no-arbitrage pricing approach, summarized in [1], uses the idea of replicating strategies to find fair prices in a simplified complete market example. This approach, however, has sharp limitations when conditions of market incompleteness are introduced. There is an extensive literature on optimal hedging in incomplete markets. We consider optimal hedging in the discrete time case. Following [2], [3], and [4], we apply the *sequential regression* approach to option hedging. We examine the question of equivalence of hedging strategies in the binomial case, and the question of stability of sequential regression under model perturbations.

Binomial Model

We introduce the *binomial asset pricing model* as presented in [1]. The binomial model consists of two components:

- A risk-free money market asset with constant interest r.
- A risky stock asset whose initial value is S_0 and whose value at each time period is determined by a "coin flip" (not necessarily fair). Given heads, the stock value will increase by an upfactor u. Given tails, the stock value will decrease by a downfactor d.

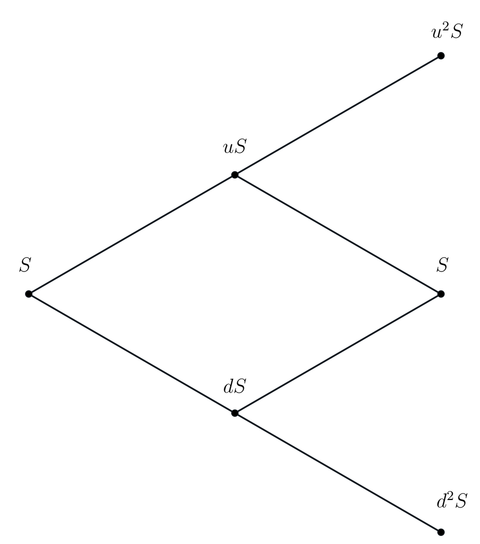

Fig. 1: Example of a 2-period Binomial Model

Given this market, we define the *wealth process* for a small investor in the market as follows:

$$X_{n+1} = (1+r)(X_n - \vartheta_n S_n) + \vartheta_n S_{n+1},$$

where X_n represents the wealth the investor has at time n and ϑ_n represents the number of stocks held at time n.

Let V_N be a random variable of the coin flips $(\omega_1, \dots, \omega_N)$ and define the stochastic process $(V_n)_{0 \le n \le N}$ such that

$$V_n(\omega_1, \dots, \omega_n) = \frac{1}{(1+r)^{N-n}} \tilde{\mathbb{E}}[V_N \mid \omega_1, \dots, \omega_n],$$

where $\tilde{\mathbb{E}}$ is the conditional expected value under risk-neutral probabilities. The binomial model represents a complete market, where there must exist a unique risk-neutral probability under which discounted option values are martingales. I.e.

$$V_0 = \frac{1}{(1+r)^N} \tilde{\mathbb{E}}[V_N].$$

We can also derive the following formula for ϑ_n from the wealth process above:

$$\vartheta_n(\omega_n) = \frac{V_{n+1}(\omega_1, \dots, \omega_n, H) - V_{n+1}(\omega_1, \dots, \omega_n, T)}{S_{n+1}(\omega_1, \dots, \omega_n, H) - S_{n+1}(\omega_1, \dots, \omega_n, T)}. \qquad (1)$$

We hereby refer to this method as *backward recursion*, and we note that it provides a unique and exact hedging strategy.

Föllmer-Schweizer Decomposition

While the method of backward recursion provides an optimal hedging strategy for the binomial model, we aim to obtain a more general method of optimal hedging. We first reframe our problem as:

$$\min_{\xi \in \Theta} \mathbb{E}[(V_N - V_0 - G_T(\xi))^2]$$

where Θ is the set of all predictable processes ξ such that $\xi_k \Delta S_k \in \mathcal{L}^2(P)$ and $G(\xi) := \sum_{j=1}^{k} \xi_j \Delta S_j$. In our particular application, V_N represents the payoff the investor must pay at time N, V_0 is the initial price of the option, or the initial capital available to the investor, and $G_T(\xi)$ is the gains the investor makes in trade. The problem is now one of minimizing net square loss, equivalent to the problems posed in [2]–[4].

If certain nondegeneracy conditions hold, we can decompose the wealth process into its almost-surely unique Doob Decomposition to obtain the *discrete Föllmer-Schweizer Decomposition*:

$$V_N = V_0 + \sum_{j=0}^{N-1} \xi_j \Delta S_{j+1} + L_N,$$

where

$$\xi_n := \frac{\mathrm{Cov}(V_N - \sum_{j=n+1}^{N-1} \xi_j \Delta S_{j+1}, \Delta S_{n+1} \mid \mathcal{F}_n)}{\mathrm{Var}(\Delta S_{n+1} \mid \mathcal{F}_n)} \quad \text{for } n = 0, \dots, N, \qquad (2)$$

where $\Delta S_n = S_n - S_{n-1}$ is the change in the stock price at time n, and L_N is a martingale with initial value 0. This hedging method is known as *sequential regression*.

Theorem. Under the binomial model, the hedging strategy $(\vartheta_n)_{0 \le n \le N-1}$ in (1) is equivalent to the hedging strategy $(\xi_n)_{0 \le n \le N-1}$ in (2).

Trinomial Model

While the binomial model is often used as an introductory tool, most models used in practice exhibit incompleteness, that is the inability to hedge exactly. An example of an incomplete market is the trinomial model. Similar to the binomial model, we have a risky asset and a risk-free asset, and the value of the risky asset at each time step is determined by a small set of outcomes. This time, we have three possible outcomes for the coin flip instead of two. That is, along with the possibility of an increase by a factor of u and decrease by a factor of d, we allow for the possibility that the stock price does not change between two consecutive time steps.

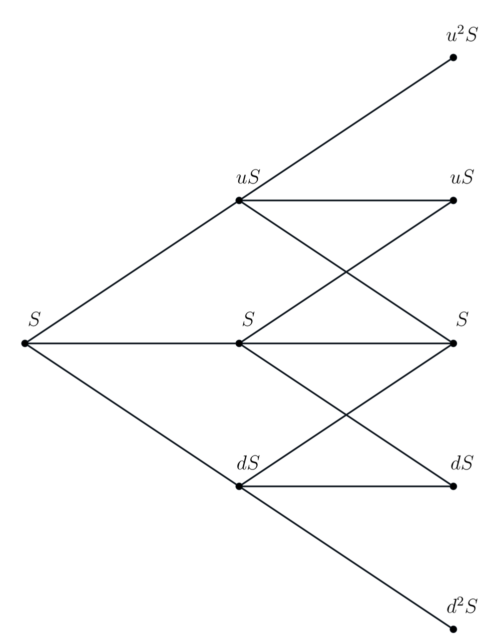

Fig. 2: Example of a 2-period Trinomial Model

If we attempt to apply backward recursion to determine a hedging strategy for the trinomial model, we obtain an overdetermined system with no solution. Thus in the trinomial model we must use the more general strategy of sequential regression to find an optimal hedging strategy.

Market Stability

We now subject the pricing model to market perturbations. We shift our focus to the change in price of the risky asset at each time step, defined by the process $(\Delta S_n)_{1 \le n \le N}$, which can be decomposed as

$$\Delta S_n = \lambda \Delta t + \sigma \Delta W$$

where λ and σ are constants describing the market, Δt is a process representing the change in time, and ΔW is a martingale with initial expectation 0 representing hedging error. Then, a market perturbation can be represented, for some $\varepsilon, \lambda', \sigma' \in \mathbb{R}$, as

$$\Delta S_n^{\varepsilon} = (\lambda + \varepsilon \lambda') \Delta t + (\sigma + \varepsilon \sigma') \Delta W$$

It is of interest to consider whether the sequential regression is stable under such perturbations; if it is not, then our predictions for asset valuation can not be considered accurate once too much error is present in our assumed values for ε, λ, and σ. Similar questions of stability were considered in a continuous time setting in [5] and [6].

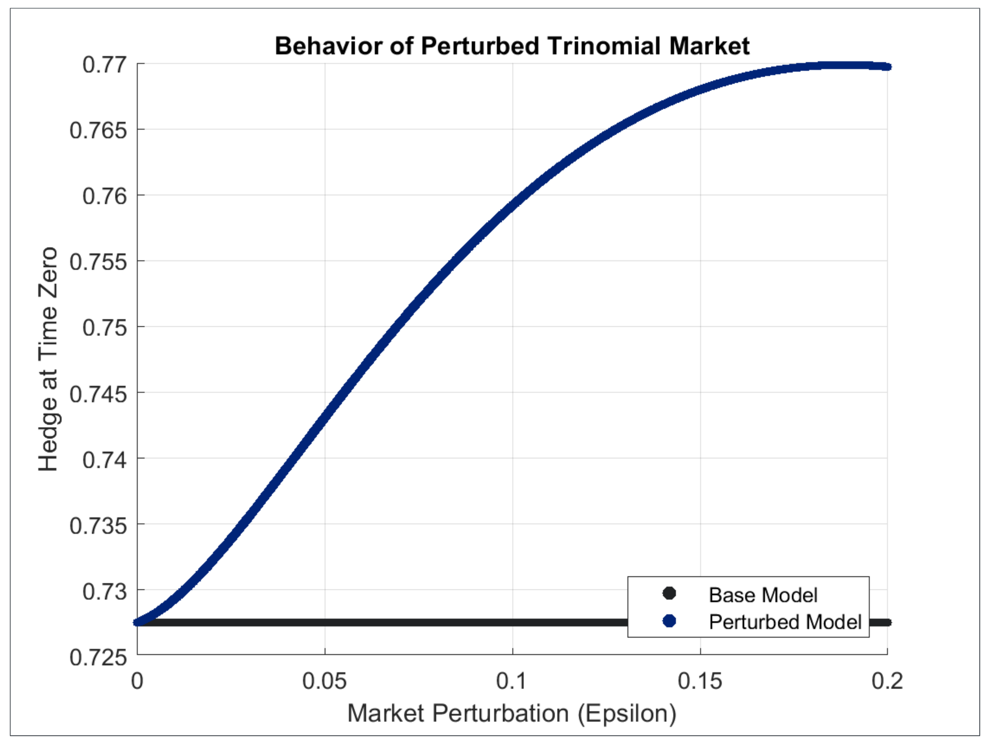

Fig. 3: Behavior of the Trinomial Model Under Market Perturbations

Theorem. In the discrete time setting, the perturbed hedge ξ^{ε} converges asymptotically to ξ^0, that is

$$\frac{\xi^{\varepsilon} - \xi^0}{\varepsilon} \xrightarrow{\varepsilon \to 0} C$$

where C is a constant dependent on market parameters.

References

[1] S. Shreve, *Stochastic calculus for finance i: The binomial asset pricing model*. Springer Science & Business Media, Nov. 6, 2012, 197 pp., Google-Books-ID: zXhCAAAAQBAJ, ISBN: 978-0-387-22527-2.

[2] H. Föllmer and M. Schweizer, "Hedging by sequential regression: An introduction to the mathematics of option trading," *ASTIN Bulletin: The Journal of the IAA*, vol. 18, no. 2, pp. 147–160, Nov. 1988, ISSN: 0515-0361, 1783-1350. DOI: 10.2143/AST.18.2.2014948.

[3] M. Schweizer, "Variance-optimal hedging in discrete time," *Mathematics of OR*, vol. 20, no. 1, pp. 1–32, Feb. 1, 1995, ISSN: 0364-765X. DOI: 10.1287/moor.20.1.1.

[4] S. Goutte, N. Oudjane, and F. Russo, "Variance optimal hedging for discrete time processes with independent increments. application to electricity markets," *ARXIV:1205.4089 [math, q-fin]*, May 18, 2012. arXiv: 1205.4089.

[5] O. Mostovyi and M. Sirbu, "Sensitivity analysis of the utility maximisation problem with respect to model perturbations," *Finance and Stochastics*, vol. 23, no. 3, pp. 595–640, 2019.

[6] O. Mostovyi, "Asymptotic analysis of the expected utility maximization problem with respect to perturbations of the numéraire," *ARXIV:1805.11427 [math]*, May 27, 2018. arXiv: 1805.11427.