Can you hear the shape of a Fractal Drum?

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7/19/2019

UCONN

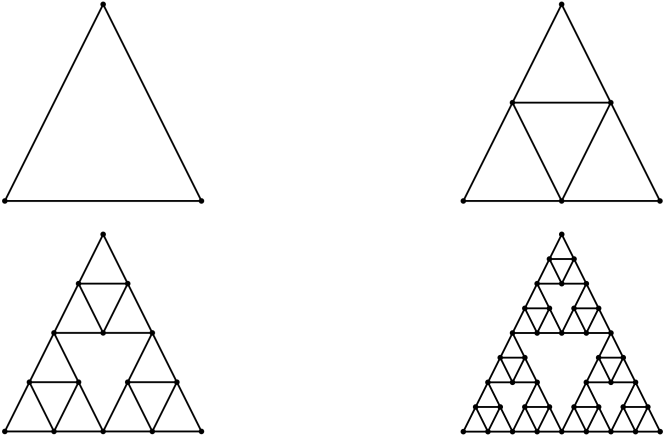
Outline

Introduction

To Analyzing Fractals

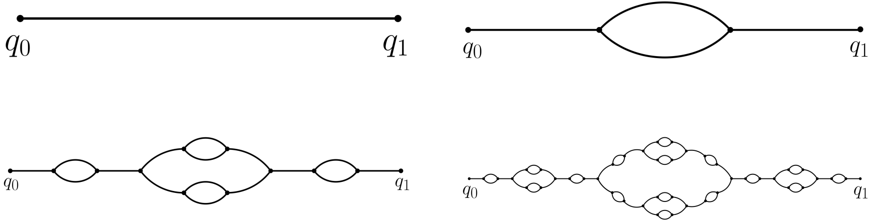
Graph Levels-For the SG

For the Sierpinski Gasket:



Graph Levels-For the Bubble Diamond

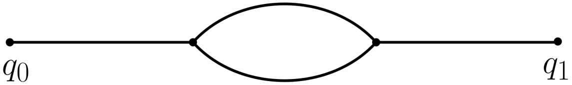
For levels of the Bubble Diamond:



Graph Levels for the Bubble Diamond-Analysis

Analyzing Level 1-The Laplacian

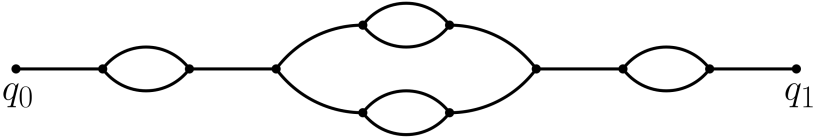
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$



$$\sigma(H) = \left\{2, \frac{4}{3}, \frac{2}{3}, 0\right\}$$

Graph Levels for the Bubble Diamond-Analysis

Analyzing Level 2



Spectral Decimation-Definition

Definition

The Laplace operator on a p.c.f. self-similar fractal G admits spectral deimation, if there exists a rational function R, a finite set A and a constant $\lambda > 1$ such that all eigenvalues of Δ can be written in the form

$$\lambda^m \lim_{n\to\infty} \lambda^n R^{(-n)}(\{\omega\}),\ \omega \in A, m \in \mathbb{N}$$

where the preimages of ω under n-fold iteration of R have to be chosen such that the limit exists. Furthermore, the multiplicities $\beta_m(\omega)$ of the eigenvalues depend only on ω and m, and the generating functions of the multiplicities are rational.

Integrated Density of States-Counting Function

Definition (Counting Function)

The Counting function is defined as $C(x) = \#\{\lambda : \lambda \leq x\}$.

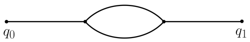
Integrated Density of States-Final Form

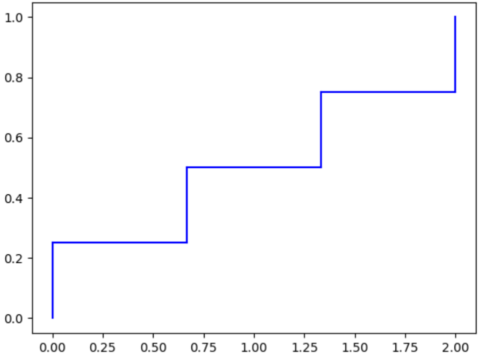
Definition (Integrated Density of States(IDS))

Let $\dim_n$ be the dimension number. The IDS is defined as $N_n(x) = \dfrac{C(x)}{\dim_n}$.

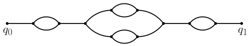
Integrated Density of States-Bubble Diamond

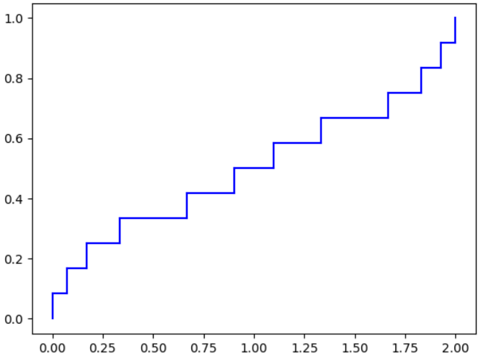
Level 1:





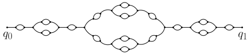
Integrated Density of States-Bubble Diamond

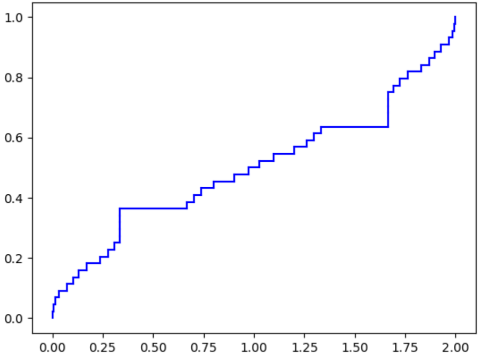




Integrated Density of States-Bubble Diamond

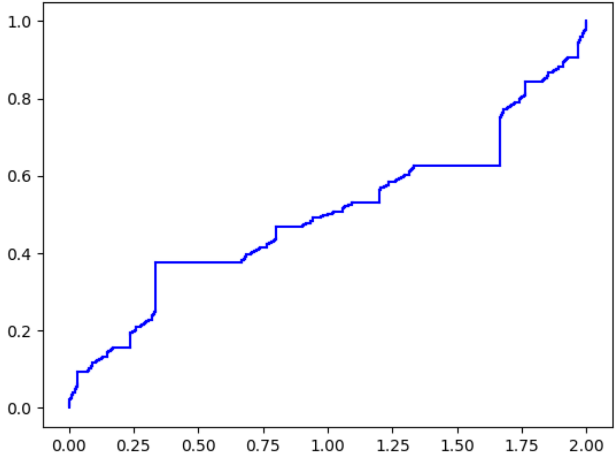
Level 3:





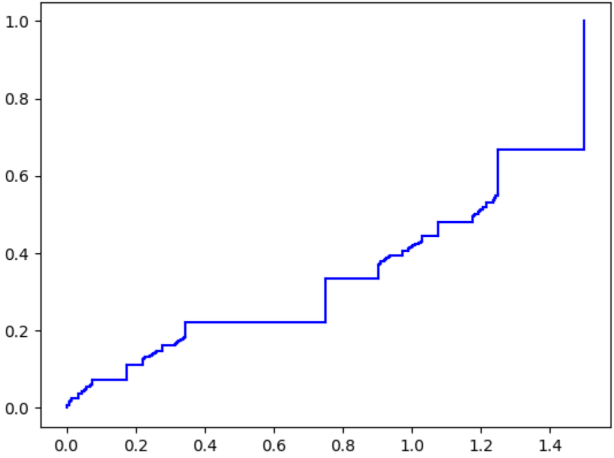
Integrated Density of States-Bubble Diamond

Level 9:



Integrated Density of States-Sierpinski Gasket

Level 10 Sierpinski Gasket:



Gap Labeling

Analyzing the Integrated Density of States

Sierpinski's Gasket

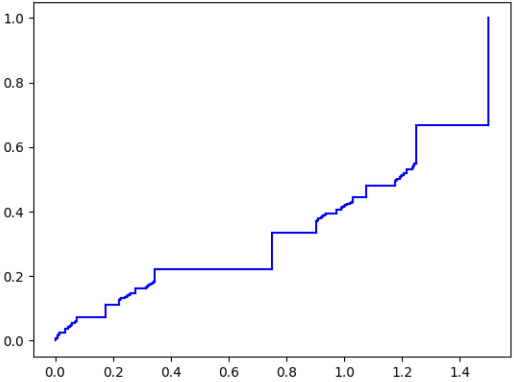


Figure 1: Level 10 N(x) for the Sierpinski's Gasket

Bubble Diamond Fractal

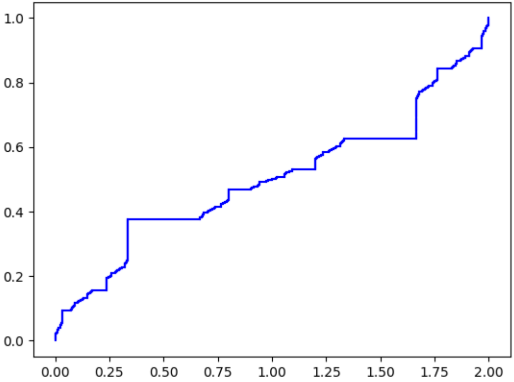


Figure 2: Bubble Diamond Level 6 IDS

Creating Formula

- We only need eigenvalue descendents of $\frac{1}{3}$ and $\frac{5}{3}$.
- The eigenvalue will have a finite number m of descendency steps before reaching $\frac{1}{3}$ or $\frac{5}{3}$.
- We add height difference between $\frac{1}{3}$ and $\frac{5}{3}$ into initial value.

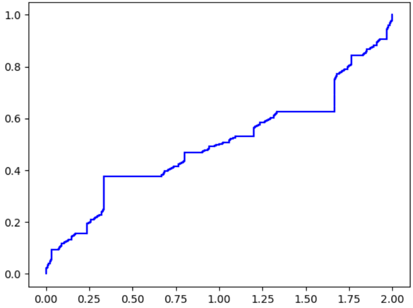


Figure 3: Bubble Diamond Level 6 IDS

The Bubble Diamond Gap Values Theorem

Theorem

The formula for the Integrated Density of States for the Bubble Diamond as follows for a chosen λ eigenvalue descendent of $\frac{1}{3}$ or $\frac{5}{3}$:

$$D_p(y) = \mathcal{I}_p + \frac{3}{2^{2m+1}}\left(x + \sum_{i=1}^{m-1} 4^{i-1}\left\lfloor \frac{x}{3^i} \right\rfloor\right)$$

where $\mathcal{I}_{\frac{1}{3}} = \frac{3}{2^{2m+3}}$ or $\mathcal{I}_{\frac{5}{3}} = \frac{5}{2^{2m+3}}$ with y being the location of that eigenvalue in the set of m descendants, $x = y - 1$, and m being the number of iterations of R^{-1} to reach the eigenvalue.

Example points

$$\text{General Case: } D_p(y) = \mathcal{I}_p + \frac{3}{2^{2m+1}}\left(x + \sum_{i=1}^{m-1} 4^{i-1}\left\lfloor \frac{x}{3^i} \right\rfloor\right)$$

Ex1. Choose Level 3 the 10th point that is a descendent of 5/3. Then the formula we will use is:

$$D_{5/3}(y) = \frac{5}{2^9} + \frac{3}{2^7}\left(9 + \sum_{i=1}^{2} 4^{i-1}\left\lfloor \frac{9}{3^i} \right\rfloor\right)$$

Ex2. Choose Level 4 the 16th point that is a descendent of 1/3. Then the formula we will use is:

$$D_{1/3}(y) = \frac{3}{2^{11}} + \frac{3}{2^9}\left(15 + \sum_{i=1}^{3} 4^{i-1}\left\lfloor \frac{15}{3^i} \right\rfloor\right)$$

The Sierpinski's Gasket Gap Values Theorem

Theorem

The formulas for the Integrated Density of States for the Sierpinski Gasket:

$$D_{\frac{3}{4},m}(y) = \frac{1}{3^{m+1}} + \frac{10}{3^{m+2}}\left(x + \sum_{i=1}^{m-1} 3^{i-1}\left\lfloor \frac{x}{2^i} \right\rfloor\right) \tag{1}$$

$$D_{\frac{5}{4},m}(y_o) = \frac{2}{3^{m+1}} + \frac{10}{3^{m+2}}\left(x + \sum_{i=1}^{m-1} 3^{i-1}\left\lfloor \frac{x}{2^i} \right\rfloor\right) \tag{2}$$

$$D_{\frac{5}{4},m}(y_e) = \frac{10}{3^{m+2}}\left(x + \sum_{i=1}^{m-1} 3^{i-1}\left\lfloor \frac{x}{2^i} \right\rfloor\right) \tag{3}$$

where for the $5/4$ *case, (2) is used if* y *is odd, and (3) is used if* y *is even.*

Significance

For the Bubble Diamond Fractal and Sierpinski's Gasket Integrated Density of States:

- The first eigenvalue has a rational height.
- The jumps between eigenvalues are rational since you can subtract the heights.
- Every eigenvalue has a rational height value since the height is given by the sum of fractions.
- The Integrated Density of States helps us analyze the frequencies on the Fractals so we can imagine that we hear the shape of a Fractal Drum.

Y'all are great

Any Questions?