Can you hear the shape of a Fractal Drum?

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Introduction
To Analyzing Fractals
Graph Levels-For the SG

For the Sierpinski Gasket:

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Graph Levels—For the Bubble Diamond

For levels of the Bubble Diamond:

$\mathcal{G}_{q_0}$ $\mathcal{G}_{q_1}$
Analyzing Level 1-The Laplacian

\[ \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix} \]

\[ \sigma(H) = \left\{ 2, \frac{4}{3}, \frac{2}{3}, 0 \right\} \]
Graph Levels for the Bubble Diamond-Analysis

Analyzing Level 2

$q_0$ $q_1$
**Definition**

The Laplace operator on a p.c.f. self-similar fractal $G$ admits spectral deimation, if there exists a rational function $R$, a finite set $A$ and a constant $\lambda > 1$ such that all eigenvalues of $\Delta$ can be written in the form

$$\lambda^m \lim_{n \to \infty} \lambda^n R(-n)(\{\omega\}), \ \omega \in A, \ m \in \mathbb{N}$$

where the preimages of $\omega$ under $n$-fold iteration of $R$ have to be chosen such that the limit exists. Furthermore, the multiplicities $\beta_m(\omega)$ of the eigenvalues depend only on $\omega$ and $m$, and the generating functions of the multiplicities are rational.
Definition (Counting Function)

The Counting function is defined as $C(x) = \#\{\lambda : \lambda \leq x\}$. 
Definition (Integrated Density of States (IDS))

Let $\dim_n$ be the dimension number. The IDS is defined as $N_n(x) = \frac{C(x)}{\dim_n}$. 
Integrated Density of States-Bubble Diamond

Level 1:

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Level 3:

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Level 9:
Integrated Density of States-Sierpinski Gasket

Level 10 Sierpinski Gasket:
Gap Labeling
Analyzing the Integrated Density of States
Sierpinski’s Gasket

Figure 1: Level 10 $N(x)$ for the Sierpinski’s Gasket

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Bubble Diamond Fractal

Figure 2: Bubble Diamond Level 6 IDS
Creating Formula

- We only need eigenvalue descendents of $\frac{1}{3}$ and $\frac{5}{3}$.
- The eigenvalue will have a finite number $m$ of descendendency steps before reaching $\frac{1}{3}$ or $\frac{5}{3}$.
- We add height difference between $\frac{1}{3}$ and $\frac{5}{3}$ into initial value.

Figure 3: Bubble Diamond Level 6 IDS
The Bubble Diamond Gap Values Theorem

**Theorem**

The formula for the Integrated Density of States for the Bubble Diamond as follows for a chosen \( \lambda \) eigenvalue descendent of \( \frac{1}{3} \) or \( \frac{5}{3} \):

\[
D_p(y) = I_p + \frac{3}{2^{2m+1}} \left( x + \sum_{i=1}^{m-1} 4^{i-1} \left\lfloor \frac{x}{3^i} \right\rfloor \right)
\]

where \( I_{\frac{1}{3}} = \frac{3}{2^{2m+3}} \) or \( I_{\frac{5}{3}} = \frac{5}{2^{2m+3}} \) with \( y \) being the location of that eigenvalue in the set of \( m \) descendants, \( x = y - 1 \), and \( m \) being the number of iterations of \( R^{-1} \) to reach the eigenvalue.
Example points

General Case: $D_p(y) = I_p + \frac{3}{2^{2m+1}} \left( x + \sum_{i=1}^{m-1} 4^{i-1} \left\lfloor \frac{x}{3^i} \right\rfloor \right)$

Ex1. Choose Level 3 the 10th point that is a descendent of $5/3$. Then the formula we will use is:

$$D_{5/3}(y) = \frac{5}{2^9} + \frac{3}{2^7} \left( 9 + \sum_{i=1}^{2} 4^{i-1} \left\lfloor \frac{9}{3^i} \right\rfloor \right)$$

Ex2. Choose Level 4 the 16th point that is a descendent of $1/3$. Then the formula we will use is:

$$D_{1/3}(y) = \frac{3}{2^{11}} + \frac{3}{2^9} \left( 15 + \sum_{i=1}^{3} 4^{i-1} \left\lfloor \frac{15}{3^i} \right\rfloor \right)$$
The Sierpinski’s Gasket Gap Values Theorem

The formulas for the Integrated Density of States for the Sierpinski Gasket:

\[
D_{3/4}^{3m}, m(y) = \frac{1}{3m+1} + \frac{10}{3m+2} \left( x + \sum_{i=1}^{m-1} 3^{i-1} \left\lfloor \frac{x}{2^i} \right\rfloor \right) \quad (1)
\]

\[
D_{5/4}^{5m}, m(y_0) = \frac{2}{3m+1} + \frac{10}{3m+2} \left( x + \sum_{i=1}^{m-1} 3^{i-1} \left\lfloor \frac{x}{2^i} \right\rfloor \right) \quad (2)
\]

\[
D_{5/4}^{5m}, m(y_e) = \frac{10}{3m+2} \left( x + \sum_{i=1}^{m-1} 3^{i-1} \left\lfloor \frac{x}{2^i} \right\rfloor \right) \quad (3)
\]

where for the 5/4 case, (2) is used if \( y \) is odd, and (3) is used if \( y \) is even.
Significance

For the Bubble Diamond Fractal and Sierpinski’s Gasket Integrated Density of States:

- The first eigenvalue has a rational height.
- The jumps between eigenvalues are rational since you can subtract the heights.
- Every eigenvalue has a rational height value since the height is given by the sum of fractions.
- The Integrated Density of States helps us analyze the frequencies on the Fractals so we can imagine that we hear the shape of a Fractal Drum.
Y'all are great
Any Questions?

The largest known prime number (as of January 2019) is $2^{82,589,933} - 1$, a number which has 24,862,048 digits when written in base 10. It was found by Patrick Laroche of the Great Internet Mersenne Prime Search (GIMPS) in 2018.

Largest known prime number - Wikipedia
https://en.wikipedia.org/wiki/Largest_known_prime_number