What can one hear about the shape of a Fractal Drum?

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Introduction

Analytic structures on fractals have been analyzed extensively in the past 50 years both because of their interesting mathematical properties and their potential applications in physics. One important question in this area is how the spectrum of a Laplacian on a fractal reflects its geometry; one version of the corresponding problem for domains in Euclidean space was famously described in Kac's question "Can you hear the shape of a drum?"[2].

Our interest is in more precise results that give the locations and multiplicities of eigenvalues explicitly. We use spectral decimation to explicitly compute the gap structure for the Laplacian on a particular two-point self-similar graph.

Laplacian

The Bubble diamond is constructed by replicating the previous stage 4 times. The first couple stages are as follows:

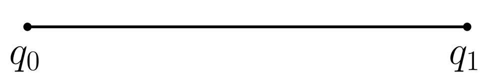

Fig. 1: The G_0 graph

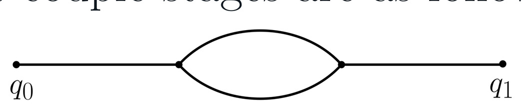

Fig. 2: The G_1 graph

The graph Laplacian is constructed from the neighbors of each point. The Level 1 graph Laplacian is below with its corresponding spectrum (list of eigenvalues).

$$\Delta_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{3} & 0 & 1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix} \qquad \sigma(\Delta_1) = \left\{2, \frac{4}{3}, \frac{2}{3}, 0\right\}$$

Spectral Decimation

Due to the properties of the Bubble Diamond, *spectral decimation* [1] can be done on the fractal, allowing us to relate the levels. More specifically, we take the preimage of a level according to a cubic polynomial $R(x)$, and then use other tools to find the rest of the eigenvalues.

Definition (*Counting Function*)**.** This is the number of eigenvalues less than or equal to x:

$$C(x) = \#\{\lambda : \lambda \le x\}.$$

Definition (*Integrated Density of States(IDS)*)**.** Observe that the number of eigenvalues at level n is dim_n.

$$N_n(x) = \frac{C(x)}{\dim_n}.$$

Where

$$N(x) = \int_0^x d\kappa = \lim_{n\to\infty} N_n(x).$$

Bubble Diamond IDS

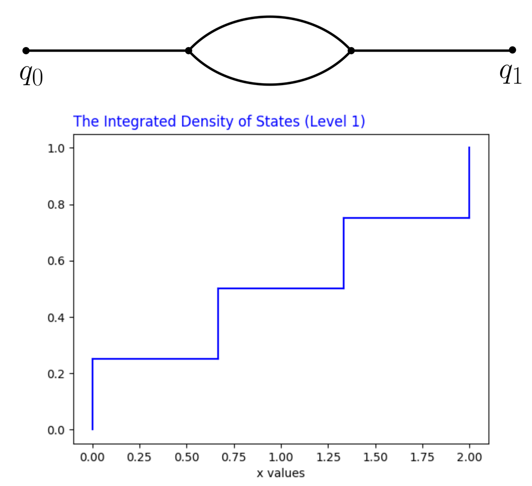

Fig. 3: G_1 graph and IDS

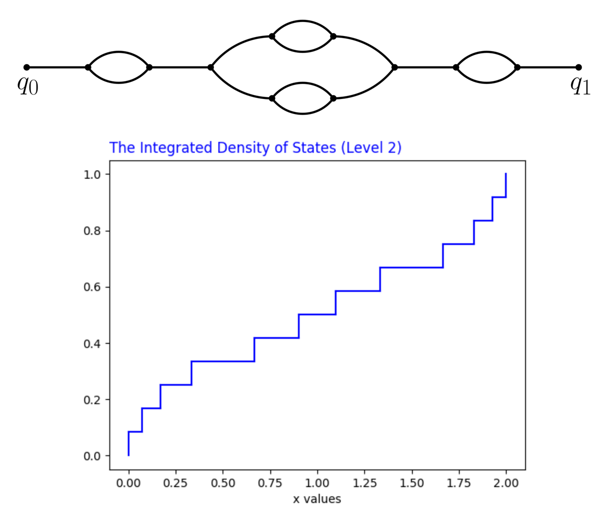

Fig. 4: G_2 graph and IDS

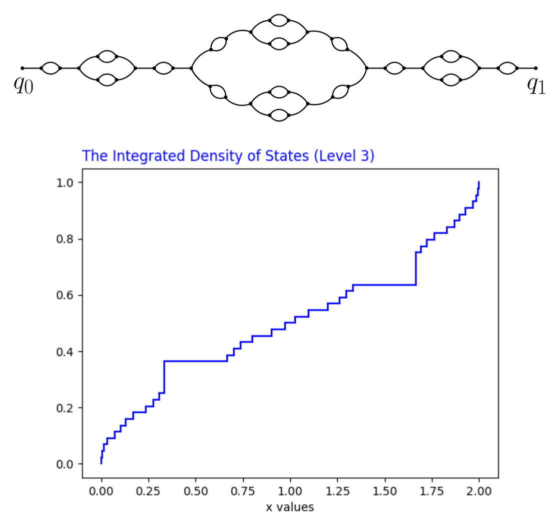

Fig. 5: G_3 graph and IDS

The height of each eigenvalue represents the proportion of the spectrum dedicated to that eigenvalues.

The picture directly to the right demonstrates our approximation of the true Bubble Diamond IDS, $N(x)$. Other patterns can be described by the following theorem:

Theorem. *The normalized limiting distribution of eigenvalues is a pure point measure κ with the set of atoms*

$$\bigcup_{m=0}^{\infty} R_{-m}\left\{\frac{1}{3}, \frac{5}{3}\right\}.$$

with weights as follows:

If $z \in R_{-m}\{\frac{1}{3}, \frac{5}{3}\}$, then

$$\kappa(z) = 2^{-2m-3},$$

Also, if $z \in \{0, 2\} \cup R_{-m}\{\frac{2}{3}, \frac{4}{3}\}$, then

$$\kappa(z) = 0.$$

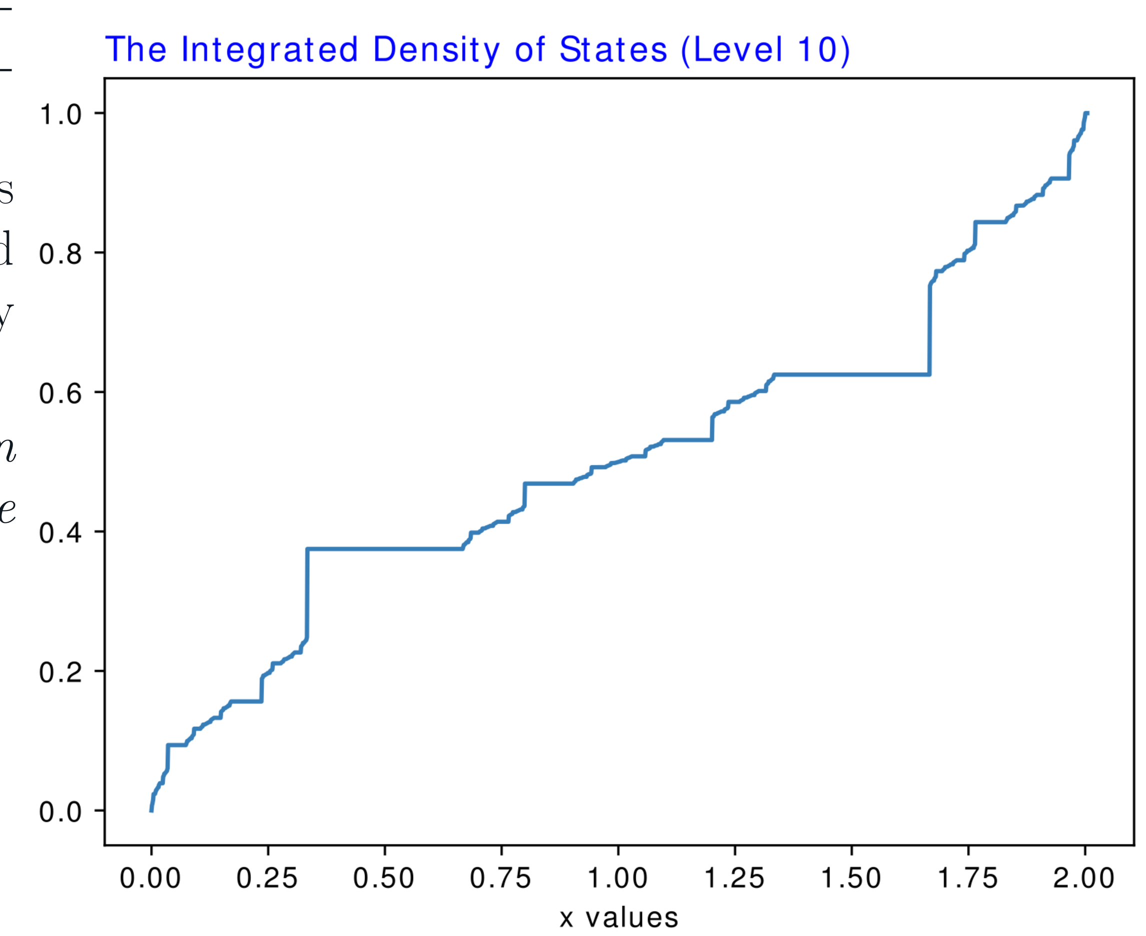

Fig. 6: Approximation of the true Bubble Diamond IDS

where $R(x) = \dfrac{z(3z-4)(3z-5)}{2}$

Convergence of Rescaled Bubble Diamond IDS

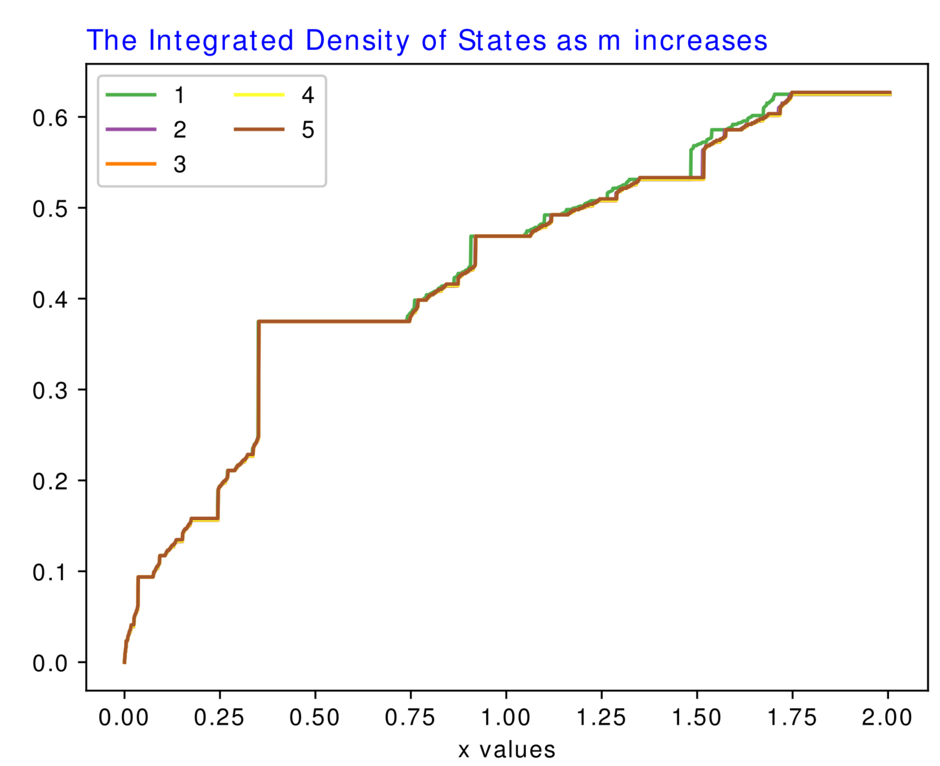

Fig. 7: Convergence with Level 10 Bubble Diamond

The Bubble Diamond graph can be modified with a certain rescaling according to a general theorem by Kigami [3]. This graph rescaling allows us to get a picture of the fractal limit. Using the scaling for the Bubble graph, the IDS is related to the spectrum of the fractal by

$$\lim_{m\to\infty} c_1^m N(c_2^m x) = \lim_{m\to\infty} 4^m N(10^{-m} x) \longrightarrow N(Q(x)),$$

where $Q(x)$ is the Poincaré map used to represent the relation.

Sierpinski Gasket IDS

A similar analysis applies to the Sierpinski Gasket.

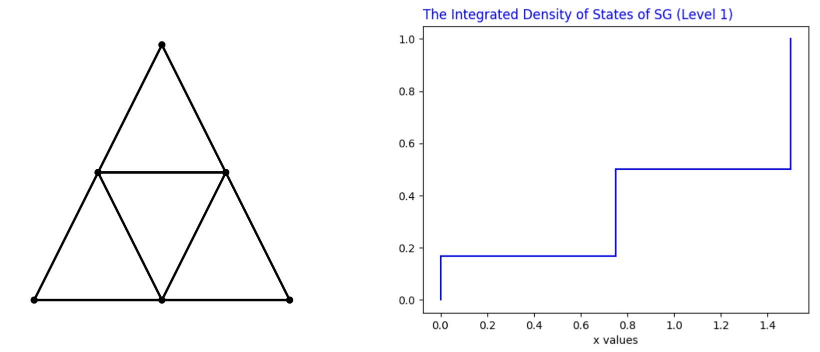

Fig. 8: G_1 graph and IDS

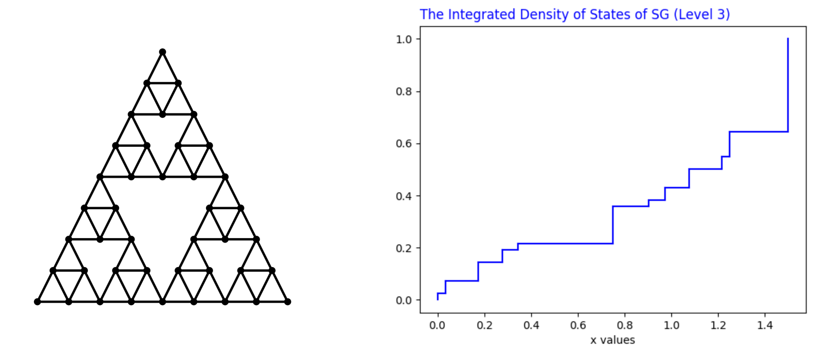

Fig. 9: G_3 graph and IDS

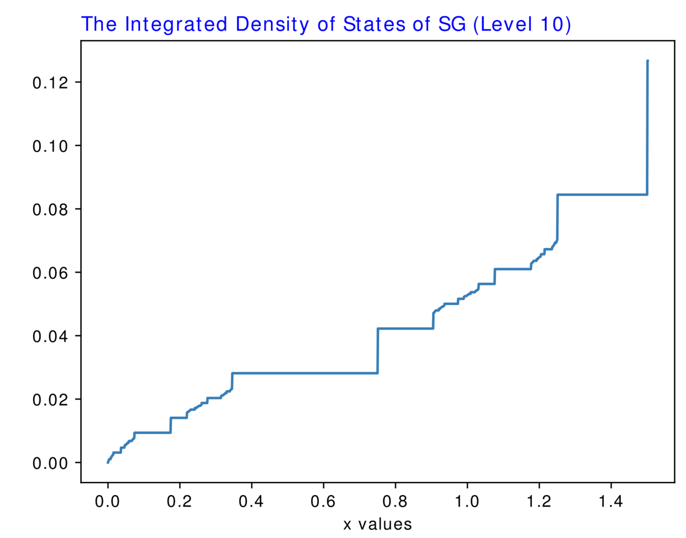

Fig. 10: Approximation of the true SG

Future Work

We have interest in the patterns of jumps because it gives us details about the topological features of the fractal. We are hoping to analyze other fractals in a similar way and test predictions made in physics literature on these fractals.

Acknowledgements

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[2] M. Kac. "Can one hear the shape of a drum?" In: *Wiadom. Mat. (2)* 13 (1971). Translated from the English (Amer. Math. Monthly **73** (1966), no. 4, part II, 1–23), pp. 11–35. ISSN: 0373-8302.

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