Gradients on Higher Dimensional Sierpiński Gaskets

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Abstract

Laplacians have been well-studied on post-critically finite (PCF) fractals. However, less is known about gradients on such fractals. Building on work by Teplyaev [1], we generalize results regarding the existence and continuity of the gradient on the standard Sierpiński Gasket to higher dimensional Sierpiński Gaskets. In particular, we find that, for functions with a continuous Laplacian, the gradient must be defined almost everywhere, and specify a set of points for which it is defined. Furthermore, we provide a counterexample on higher-dimensional Sierpiński gaskets where the Laplacian is continuous but the gradient is not defined everywhere. We conjecture that Hölder continuity of the Laplacian is a condition strong enough to guarantee that the gradient exists at each point.

Introduction

The Sierpiński Gasket is among the simplest of post-critically finite (PCF) fractals. As a result, it can be a very effective testing-ground for many concepts related to analysis on fractals. In particular, we are interested in conditions that allow for the existence and continuity of ∇u, where u is in the domain of the Laplacian.

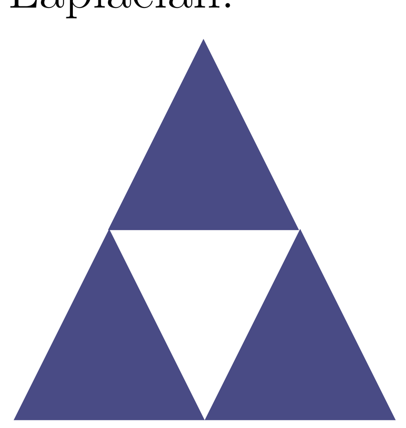

Figure 1: SG_3 first level

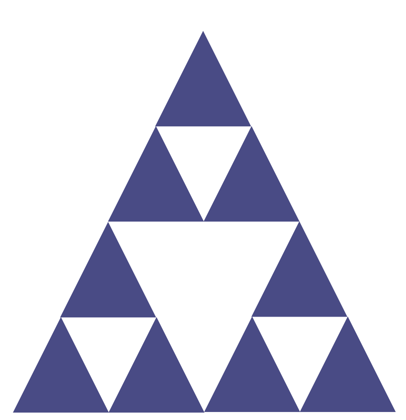

Figure 2: SG_3 second level

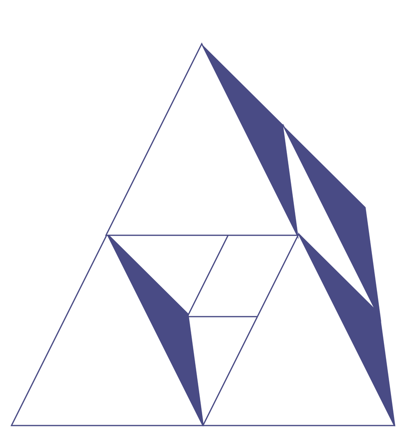

Figure 3: SG_4 first level

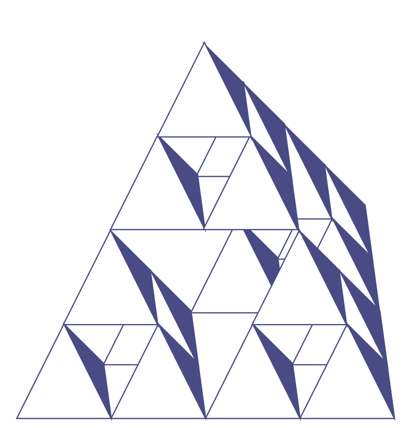

Figure 4: SG_4 second level

What is the Higher-Dimensional Sierpiński Gasket?

Given a set of points $\{p_i\}_{i=0}^{N-1} \subseteq \mathbb{R}^{N-1}$ that define the vertices of a regular $N-1$-simplex, we construct an iterated function system $\{F_i\}_{i=0}^{N-1}$ where $F_i(x) = \frac{1}{2}(x-p_i)+p_i$ for $x \in \mathbb{R}^{N-1}$. The Sierpiński Gasket is defined as the unique compact set SG_N such that $SG_N = \cup_{i=0}^{N-1} F_i(SG)$. See Figures 3 and 4 for a depiction of SG_4. Each point in SG_N can be defined by at least one infinite word $\omega \in \Omega$, where $\Omega = \{0, \cdots, N-1\}^\infty$.

Laplacian and Gradient

Given a probability measure μ on a PCF fractal K, we can define the Laplacian operator Δ_μ. This is an analogue to the second derivative, and is in fact equal to the second derivative on the unit interval with Lebesgue measure. We define harmonic functions as continuous functions satisfying $\Delta_\mu h = 0$; on the unit interval, these are linear functions. The gradient $\nabla u(\omega)$ is defined as the limit of secants $\nabla_n u(\omega)$, which are harmonic functions that agree with u on the boundary of an n-cell containing the point specified by ω. In [1], Teplyaev provides a condition for the pointwise existence of the gradient, which we use as the basis for all arguments concerning differentiability.

Results

Asymmetric Bernoulli Measures

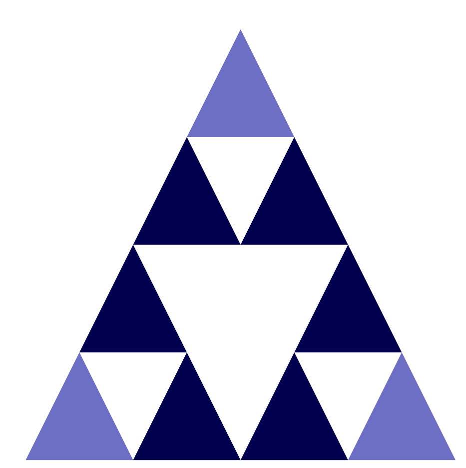

Figure 5: SG_3 first iteration

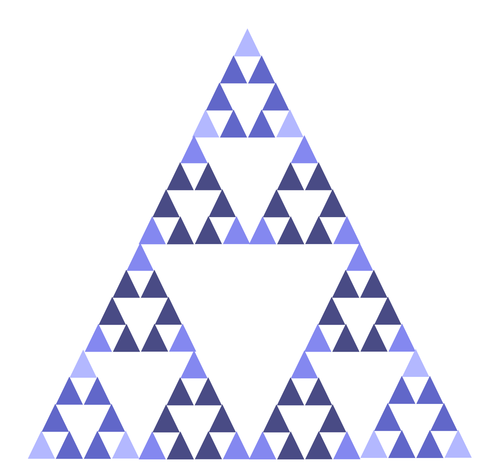

Figure 6: SG_3 second iteration

By considering asymmetric Bernoulli measures μ that satisfy Teplyaev's condition, we produce a family of examples where, for each u in the domain of Δ_μ, u has a continuous gradient. Let $\{\mu_{ij}\}$ be the set of measure weights for measure μ on SG_3. The measure μ satisfies Teplyaev's condition if

$$\mu_{ij} < \begin{cases} \frac{1}{9} & i = j \\ \frac{1}{\sqrt{17+4\sqrt{13}}} & i \neq j \end{cases}.$$

We depict this graphically in Figure 5, with the darker cells representing those with greater measure. Figure 6 represents the second iteration of this Bernoulli measure.

The following theorem is a generalization of a result of Teplyaev to higher dimensional Sierpiński Gaskets equipped with the standard Bernoulli measure.

Theorem 1

Let $u\colon SG_N \to \mathbb{R}$ and suppose Δu is continuous. Then $\nabla u(\omega)$ is defined at every $\omega \in \Omega$ such that:

$$\liminf_{n\to\infty} \frac{C_N(\omega, n)}{\log n} \geq \gamma,$$

where $\gamma > 0$ is a certain constant.

Here, $C_N(\omega, n)$ refers to a counting function defined in [3]. In fact, we can show that the subset of words that violate the condition above has measure zero with respect to the standard Bernoulli measure in Ω.

Counterexample Function

In Theorem I, we assumed that $\Delta_\mu u$ is continuous. Is continuity of $\Delta_\mu u$ enough to guarantee the existence of the gradient for all $\omega \in \Omega$? To show this is not the case, we constructed a counterexample function u whose Laplacian is continuous, although the gradient is not defined along an edge of SG_4.

The following conjecture, based on a result for SG by Teplyaev [1, Thm. 3], provides a stronger condition for the higher-dimensional gasket which guarantees the existence of the gradient for each $\omega \in \Omega$.

Conjecture

Let $u : SG_N \to \mathbb{R}$, and let μ be the standard measure on SG_N. If $\Delta_\mu u$ is Hölder continuous, then $\nabla u(\omega)$ exists for all $\omega \in \Omega$.

Future Objectives

We are currently in the process of proving the previous conjecture. In addition to this, we are interested in examining the use of Kusuoka measure, rather than Bernoulli measure.

References

[1] Alexander Teplyaev. Gradients on Fractals. Journal of Functional Analysis, Elsevier, 1998. McMaster University.

[2] Robert Strichartz. Differential Equations on Fractals. Princeton University Press, Princeton, NJ, 08540, 2006. Cornell University.

[3] Luke Brown, Giovanni Ferrer, Luke Rogers, Karuna Sangam, Gamal Mograby. Gradients on Higher Dimensional Sierpiński gaskets. 2017. University of Connecticut.

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