# Geodesic Interpolation on Sierpinski Simplices

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## Introduction

In Euclidean space, two sets may be interpolated along straight lines connecting all pairs of points in the two sets. In more general spaces, interpolation happens along geodesics—shortest paths parameterized at unit speed. We study interpolation on Sierpinski simplices, which generalize the well-known Sierpinski triangle. In addition to finding an upper bound on the number of geodesics, we show some interesting self-similarity properties of interpolant measures, and prove an analogue of the classical Brunn–Minkowski inequality for interpolant sets.

## The Sierpinski $n$-Simplex

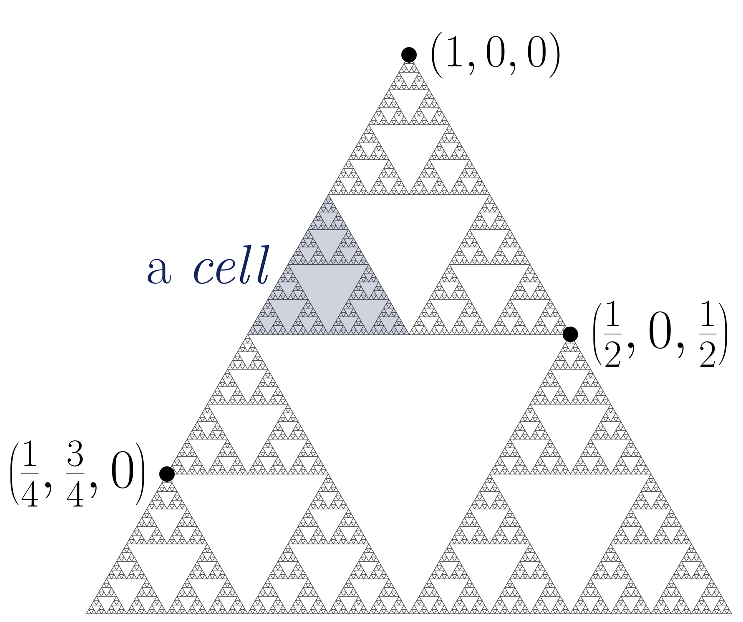
The *Sierpinski $n$-simplex* $S_n \subseteq \mathbb{R}^n$ is the attractor of the iterated function system (IFS)

$$F_i : \mathbb{R}^n \to \mathbb{R}^n, \quad F_i(x) = \tfrac{1}{2}(x + q_i),$$

for $\{q_0, \ldots, q_n\}$, the vertices of a unit $n$-simplex. Any point $x \in S_n$ can be expressed uniquely as a convex combination of $q_0, \ldots, q_n$:

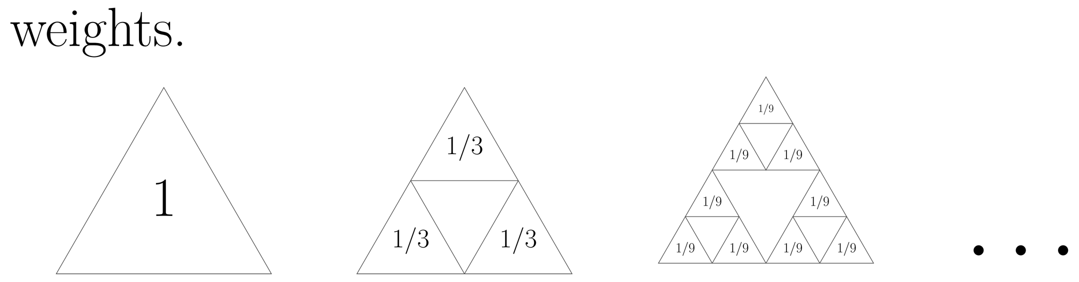
$$x = c_0 q_0 + \cdots + c_n q_n.$$

$(c_0, \ldots, c_n)$ are the *barycentric coordinates* of $x$, and we denote $c_i$ by $[x]_i$.




*$S_2$, with barycentric coordinates of highlighted points.*

$S_n$ has no volume in $\mathbb{R}^n$, so we measure sets using a self-similar measure $\mu_n$ defined by $n+1$ equal weights.



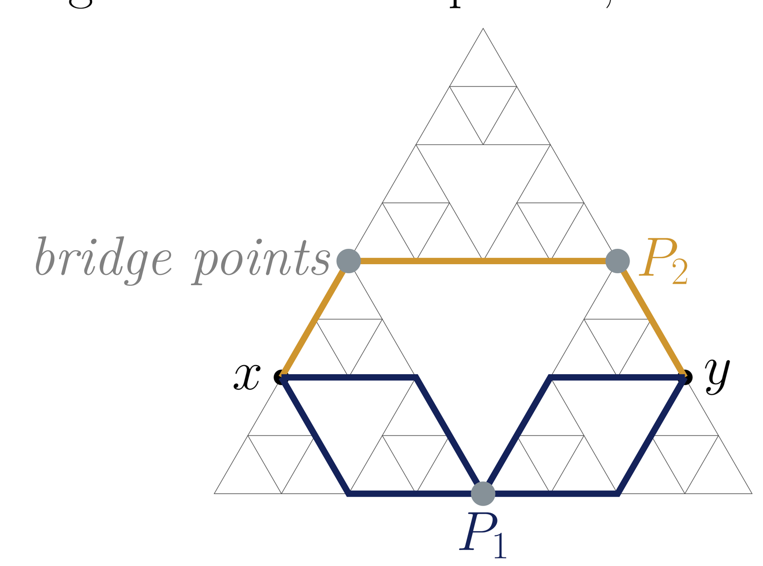

*The measure $\mu_2$ of cells in $S_2$.*

## Geodesics

The *intrinsic metric* $d(x, y)$ is defined to be the length of a geodesic from $x$ to $y$. The self-similarity and symmetry of $S_n$ relate barycentric coordinates to geodesic distance:

***Proposition.*** *Let $x$ be a point and $y$ the boundary point of its cell. Then $d(x, y) = [x]_i - [y]_i$.*

To address geodesic non-uniqueness, we define:

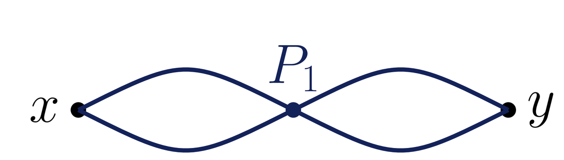



*$P_1$ paths pass through the common bridge point of the cells containing $x$ and $y$; $P_2$ paths do not.*

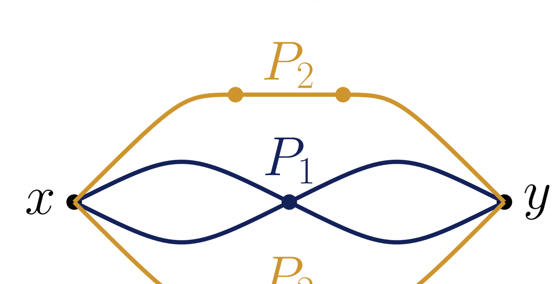
For points $x, y \in S_n$, we use the following lemmas in the proof of the theorem below:

- There are at most two geodesics between $x$ and a boundary point of its containing cell.
- There exist $P_2$ paths between $x$ and $y$ passing through at most two pairs of bridge points.
- If there exist $P_2$ paths between $x$ and $y$ through two pairs of bridge points, there are exactly two $P_2$ paths from $x$ to $y$.

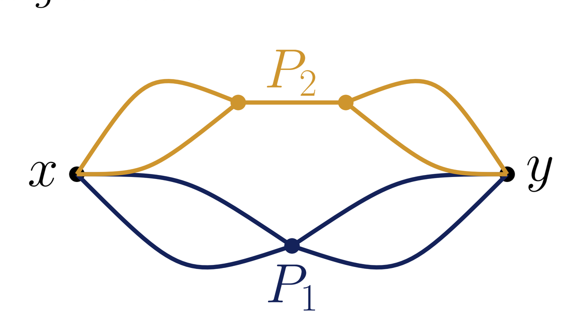
***Theorem.*** *There exist at most eight distinct geodesics between any two points in $S_n$.*




(a) *4 $P_1$ paths and no $P_2$ paths—4 total geodesics.*




(b) *4 $P_1$ paths and 2 $P_2$ paths—6 total geodesics.*




(c) *4 $P_1$ paths and 4 $P_2$ paths—8 total geodesics.*

## Cell-to-point Interpolation

Let $A, B \subseteq S_n$. We define the interpolant $Z_t : A \times B \to S_n$ for all $t \in [0, 1]$ as

$$Z_t(a, b) = \hat{\gamma}_{a \to b}(t),$$

where $\hat{\gamma}_{a \to b}$ is a geodesic from $a$ to $b$.
We want to study the "density" of the interpolant set when $A$ is a cell and $B$ is a single point $b$, so we define a pullback measure on the common path:

$$\eta_t(X) = \mu_n(Z_{t,b}^{-1}(X)) \quad \text{for all } t.$$

The connection between geodesic distance and barycentric distance means that points with equal $i$-th barycentric coordinate coincide when interpolated. So the interpolant density is essentially a projection of the measure $\mu_n$; we call the projection $\nu_n$.

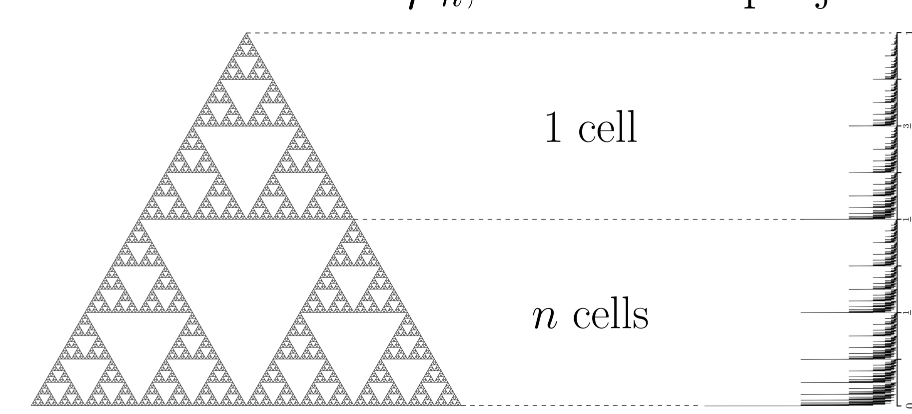



*Illustration of the projection measure $\nu_n$ for a cell in $S_n$.*

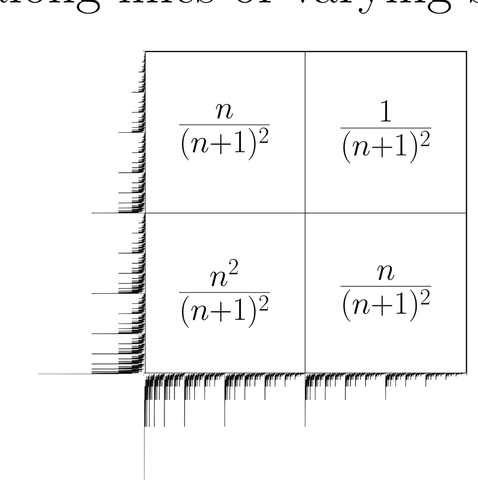
***Theorem.*** *The measure $\eta_t$ is self-similar, with weights $\frac{1}{n+1}$ and $\frac{n}{n+1}$.*

## General Interpolation

When $A$ and $B$ are cells, we define a pullback measure on the common path:

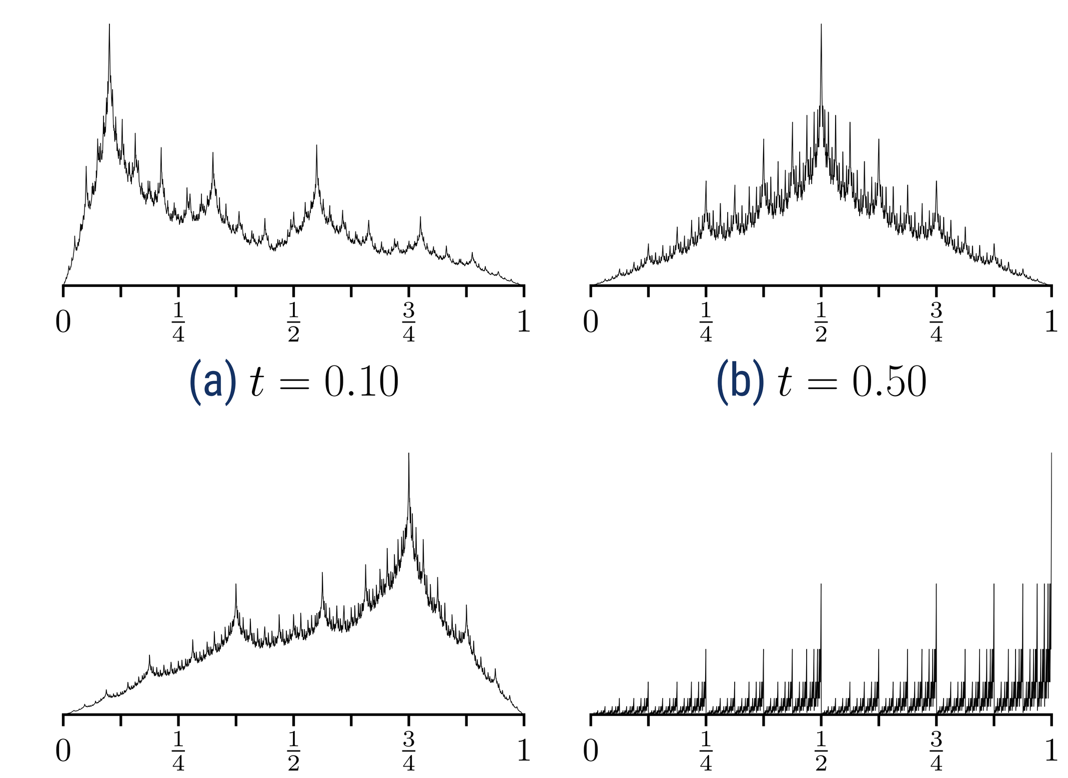
$$\eta_t(X) = \mu_n \times \mu_n(Z_t^{-1}(X)).$$

The same barycentric results as above indicate that we can view $\eta_t$ as the projection of the product measure $\nu_n \times \nu_n$ along lines of varying slope.




*Schematic of the product measure $\nu_n \times \nu_n$.*

Since $\eta_t$ is a projection of a self-similar product measure, it is also self-similar. The changing slope of the projection over time means that the self-similarity changes as well:

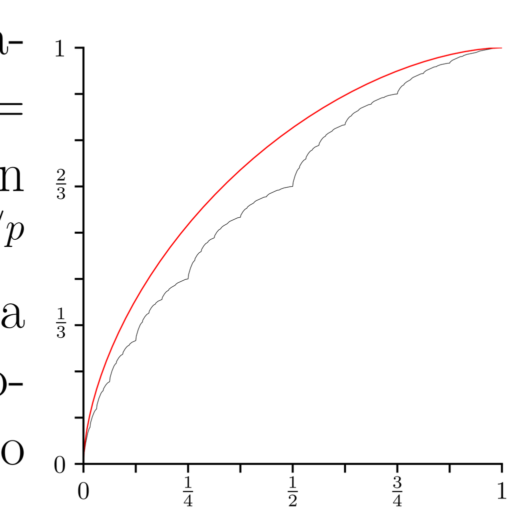



(a) $t = 0.10$ (b) $t = 0.50$ (c) $t = 0.75$ (d) $t = 1.00$

*Density of $\eta_t$ at various values of $t$.*

## Interpolation Inequalities

We can bound the cumulative measure function $g(x) = \nu_n([0, x])$ by the function $\Phi(x) = (1 - (1 - x)^p)^{1/p}$ using self-similarity, for a sufficiently large $p$. Substituting this bound into the one-dimensional Brunn–Minkowski inequality on the common path yields our interpolation inequality.




*Bounding $g(x)$.*

***Theorem.*** *Let $A, B \subseteq S_n$ be disjoint connected sets. Then for all $t \in (0, 1)$,*

$$1 - (1 - \mathcal{H}^1(Z_t(A, B)))^{d_n} \geq (1-t) \cdot \mu_n(A)^{d_n} + t \cdot \mu_n(B)^{d_n},$$

*where $d_n = \frac{\log \frac{1}{2}}{\log \frac{n}{n+1}}$.*

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