Computations on the Koch Snowflake

Carlos D. Lima, Malcolm H. Gabbard, Gamal Mograby, Luke Rogers, Alexander Teplyaev

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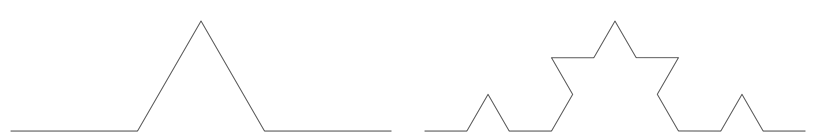
Abstract

A Laplacian is applied to graph approximations of the Koch Snowflake. Numerical approximations indicate a localization of the Laplacian eigenfunctions at high energies.

Constructing the Snowflake

The **Koch Curve** K is approximated by a sequence of finite graphs, K_n [4]:



(a) K_1 (b) K_2

Figure 1: First and Second Level Approximations

The **Koch Snowflake**, denoted $\bar{\Omega}$, is given by taking a union of three copies the **Koch Curve** K.

Energies on $\bar{\Omega}$

We consider a graph Laplacian L generated by the following Dirichlet energy form:

$$E_n(f) = \sum_{p_1, p_2 \epsilon V_n} c_n(p_1, p_2)(f(p_1) - f(p_2))^2$$

where c_n for two points p_1, p_2 on $\bar{\Omega}$ is,

$$c_n(p_1, p_2) = \begin{cases} 1 & \text{if } p_1, p_2 \text{ share an interior edge} \\ 4^n & \text{if } p_1, p_2 \text{ share an outer boundary edge} \\ 0 & \text{if } p_1, p_2 \text{ not connected by an edge.} \end{cases}$$

This Laplacian form is defined w.r.t. the measure,

$$m_n(p) = \begin{cases} \frac{1}{9^n} & \text{if } p \text{ is an interior vertex} \\ \frac{1}{4^n} & \text{if } p \text{ is an outer boundary vertex} \end{cases}$$

Eigenfunctions of L on $\bar{\Omega}$

Results from [3] were reproduced by imposing Dirichlet B.C.:

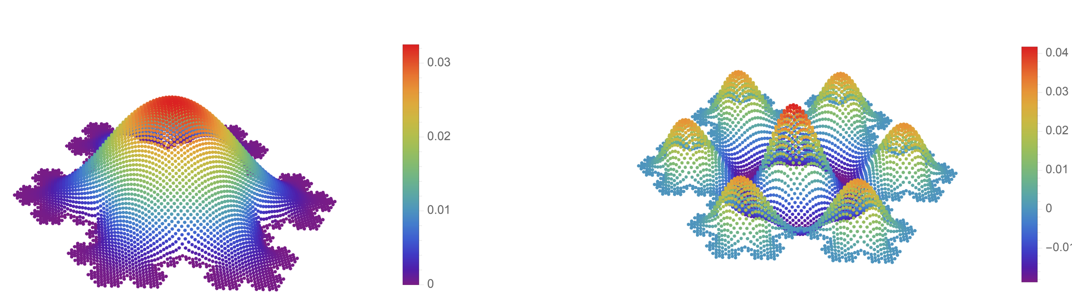
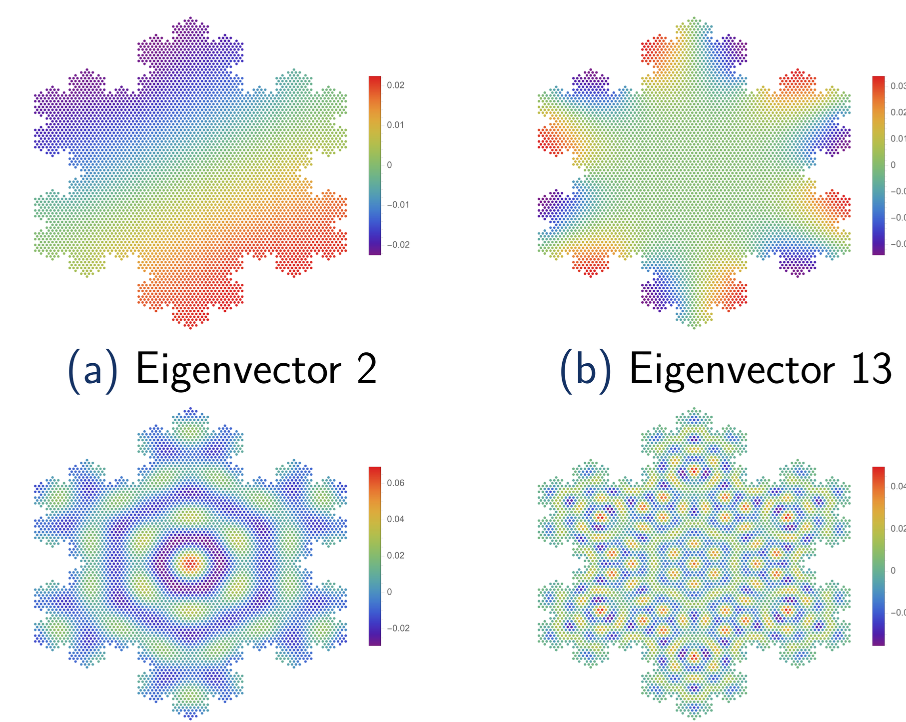


Figure 2: First (left) and Thirteenth (right) eigenvectors with Dirichlet B.C.

The eigenvalues and eigenvectors of L change drastically without Dirichlet B.C.:



(a) Eigenvector 2 (b) Eigenvector 13

(c) Eigenvector 125 (d) Eigenvector 579

Figure 3: Contour plots of eigenvectors without Dirichlet B.C.

Localization of Energy

Motivated through experiments by Sapoval[1] where they showed focused localization for a similar fractal, the question was posed as to whether there existed localization for eigenfunctions of L. Approximations indeed indicate a form of localization on $\bar{\Omega}$, unlike the results from [3] with Dirichlet B.C.

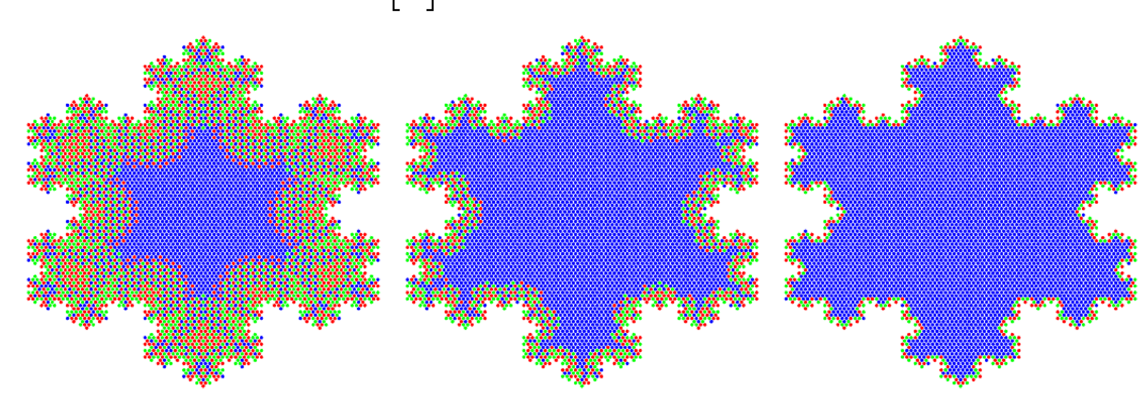
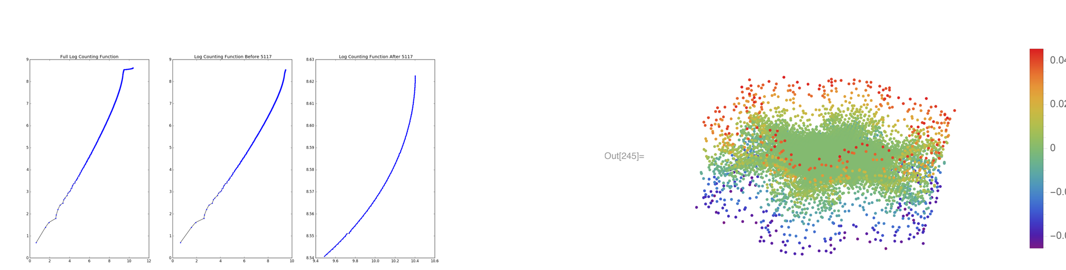


Figure 4: First sign of energy localization to the boundary of $\bar{\Omega}$ (left) and a continued "zeroing out" of the inner region(middle and right) at higher energies.

Features of Localization

The counting function (a) gives a characterizing feature, a kind of inflection point, indicating a localization of energy to the boundary of $\bar{\Omega}$.



(a) Counting Function of eigenvectors. (b) Eigenvector 5117

Figure 5: Localization of Eigenvectors on $\bar{\Omega}$

The "zeroing out" of the inner region may be due to the high-frequency oscillations happening on the boundary of $\bar{\Omega}$.

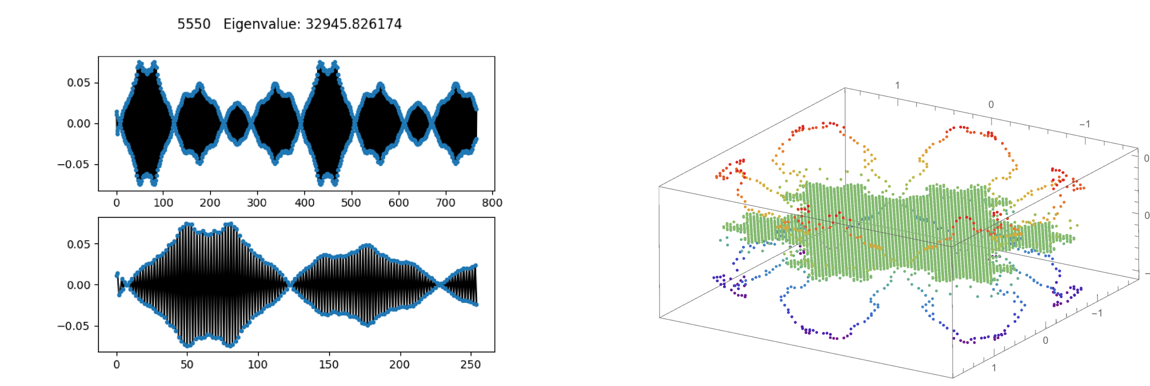


Figure 6: 2D plot of boundary for eigenvector 5550

The localization on $\partial\Omega$ is qualitatively similar to high-frequency localization seen in whispering gallery modes. A landscape function, in the style of Filoche and Mayboroda [2], is generated and correctly predicts where eigenfunctions localize, seen in Figure 7.

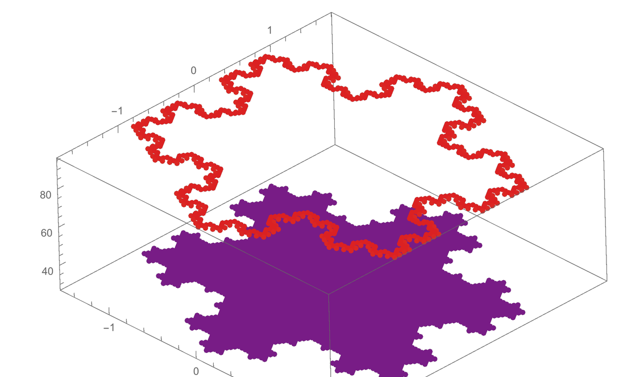


Figure 7: Landscape Function

Future Work

A more robust characterization of this high-frequency localization is needed, and more mathematically rigorous formulations are under way.

Acknowledgements

[1]:Sapoval, *Experimental Observation of Local Modes in Fractal Drums*,(Physica D, France, 1989)

[2]:M. Filoche, S. Mayboroda *The Hidden Landscape ofLocalization*, (Ecole Polytechnique X, 2011)

[3]: Lapidus, *Snowflake Harmonics and Computer Graphics*, (U.C. Riverside, 1995)

[4]: Frieberg, *Energy Form on a Closed Fractal Curve*, Z. Anal. Anwend (2004)

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