Lipschitz Continuity on PCF Fractals

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Motivating Question

The Sierpinski Gasket (SG) is a classical example of a self-similar fractal, and has been well studied. What has not been as well studied is the harmonic Sierpinski Gasket ($SG_{\mathcal{H}}$) (see Figure 2), which is the embedding of SG into $\mathbb{R}^2$. The goal for our project was to better understand $SG_{\mathcal{H}}$ by proving that Laplacian eigenfunctions on $SG_{\mathcal{H}}$ are Lipschitz continuous.

PCF Self-Similar Sets

A **self-similar** set, K, is defined using an iterated function system $F_1, \dots, F_N$ of contractions on a complete metric space (X, d), where K is the unique non-empty compact fixed point of this iterated function system, i.e. $K = \bigcup_{i=1}^{N} F_i K$. (see Figure 1)

For a word $w = w_1 w_2 \cdots w_n$ with $w_j \in \{1, \dots, N\}$ we define the map $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}$.

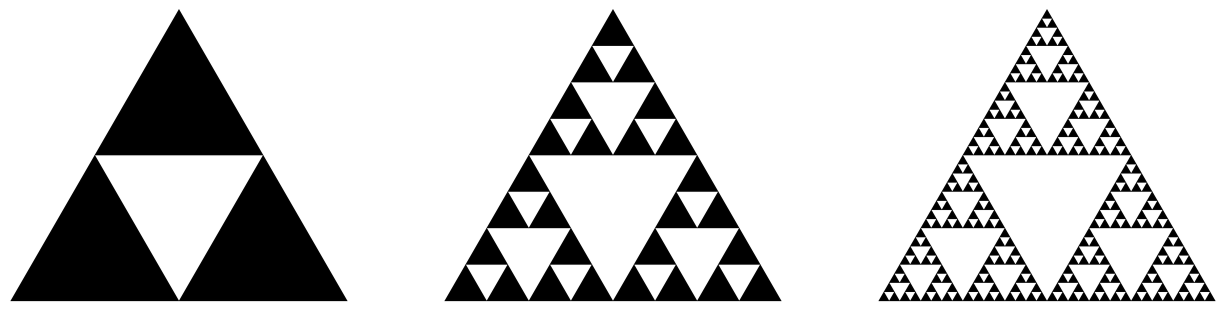

Figure 1: First, third, and fifth iteration of the Sierpinski gasket iteration

A **post-critically finite** (p.c.f) self-similar set is one for which there exists a finite set $V_0 \subset K$, such that for words $w \neq w'$ and $|w| = |w'|$, $F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0$. We call V_0 the boundary of K. Further, we require that each boundary point be a fixed point of some F_i. From V_0, we define $V_m = \cup_{|w|=m} F_w V_0$ and $V_* = \cup_m V_m$.

Regular Harmonic Structure

A **regular harmonic structure** consists of a vector of resistance scalings $(r_1, \dots, r_N)$, with each $r_i \in (0, 1)$, and an energy form on functions on V_0 denoted $\mathcal{E}_0(u, v)$. From $\mathcal{E}_0$, we can iteratively define an energy $\mathcal{E}_m$ on V_m, and finally arrive at a resistance form on functions on V_* by letting
$\mathcal{F} = \{u : \lim_{m\to\infty} \mathcal{E}_m(u, u) < \infty\}$ and $\mathcal{E}(u, v) = \lim_{m\to\infty} \mathcal{E}_m(u, v)$ for $u, v \in \mathcal{F}$.
We call a function **harmonic** if it minimizes $\mathcal{E}_m$ for all $m \geq 1$ given values on V_0. From these we construct the harmonic spline ψ_p which is 1 at $p \in V_1 \setminus V_0$ and zero otherwise on V_1.

Laplacian and Green's Function

Let μ be a Borel regular probability measure on K. We define a **Laplacian** on K as follows: For $u \in \mathcal{F}$ and $f \in L^p(\mu)$, we say that $u \in \operatorname{dom}_{L^p} \Delta_\mu$ and $\Delta_\mu u = f$ if

$$\mathcal{E}(u, v) = -\int_K f v \, d\mu \text{ for all } v \in \mathcal{F} \text{ with } v|_{V_0} = 0$$

We also have a Green's operator G which inverts the Laplacian, and an associated **Green's function** with the form

$$g(x, y) = \sum_{m=0}^{\infty} \sum_{|w|=m} r_w \sum_{p,q \in V_1 \setminus V_0} h_{p,q} \, \psi_p(F_w^{-1}(x)) \, \psi_q(F_w^{-1}(y)),$$

where $r_w = r_{w_1} \dots r_{w_n}$.

Results

Theorem: Suppose K is a p.c.f. self-similar set in (X, d) and supports a regular harmonic structure such that the functions F_j^{-1} are Lipschitz and the harmonic splines ψ_p are α-Hölder continuous. Let L_w be the Lipschitz constant of F_w^{-1} and define

$$\Psi_\alpha(x, y) = \sum_{m=0}^{\infty} \sum_{|w|=m} r_w L_w^\alpha \mathbb{1}_{F_w K}(x) \mathbb{1}_{F_w K}(y),$$

where $\mathbb{1}_{F_w K}$ is the indicator function on $F_w K$.
If $u \in \operatorname{dom}_{L^p} \Delta_\mu$ and $\Psi_\alpha(x, \cdot) \in L^{p'}(\mu)$ uniformly in x for conjugate exponents p and p', then u is α-Hölder continuous with

$$|u(x) - u(x')| \leq C \|\Delta_\mu u\|_p \sup_{x \in K} \|\Psi_\alpha(x, \cdot)\|_{p'} d(x, x')^\alpha.$$

Corollary: If K is as in the theorem above, and $\Psi_\alpha(x, y)$ is in $L^1(d\mu(y))$ uniformly with respect to x then Laplacian eigenfunctions and the heat kernel are α-Hölder.
Method: Since G inverts the Laplacian, it follows that $G\Delta_\mu u = u$. Thus, we show the α-Hölder continuity of u by showing that $G\Delta_\mu u$ is α-Hölder continuous. The advantage of this method is that it allows us to make use of the Green's function.

Conclusion

Regarding our motivation, we were able to partially answer our motivating question by proving that L^p Laplacian functions of $SG_{\mathcal{H}}$ are α-Hölder continuous. However, we were able to go beyond our initial goals by proving a result for a broad class of p.c.f. self-similar sets beyond $SG_{\mathcal{H}}$, such as the unit interval and the Sierpinski gasket.

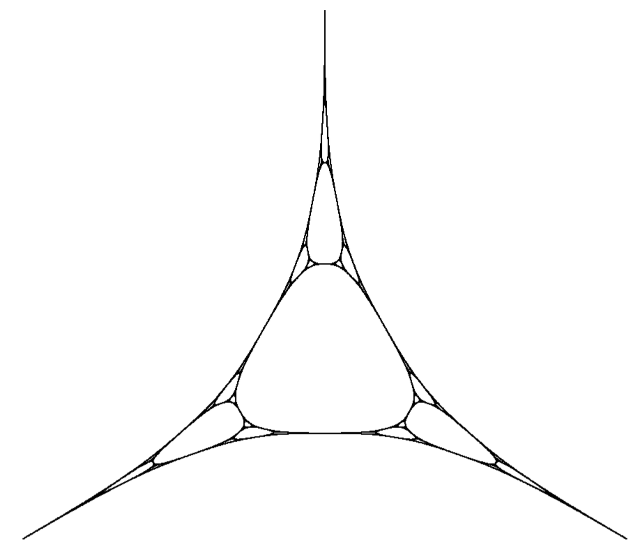

Figure 2: Harmonic Sierpinski Gasket

Future Research

We hope to extend our technique to prove Lipschitz continuity for Laplacian eigenfunctions on $SG_{\mathcal{H}}$ by applying G^2 or G^3 to Δ_μ. We also hope to generalize our current results to a class of finitely ramified self-similar sets, or to general self-similar sets expressed by harmonic coordinates.

References

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