Applications of Multiplicative LLN and CLT for Random Matrices

Rajeshwari Majumdar, Phanuel Mariano, Hugo Panzo, Lowen Peng, and Anthony Sisti

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Abstract

The Lyapunov exponent is the exponential growth rate of the operator norm of the partial products of a sequence of independent and identically distributed random matrices. It usually cannot be computed explicitly from the distribution of the matrices. Furstenberg and Kesten (1960) and Le Page (1982) found analogues to the Law of Large Numbers and Central Limit Theorem, respectively, for the norm of the partial products of a sequence of such random matrices. We use these analogues to efficiently compute the Lyapunov exponent for several random matrix models and numerically estimate the corresponding variances. For random matrices of order 2, with independent components distributed as Bernoulli $\left(\frac{1}{2}\right)$, we obtain analytic estimates for the Lyapunov exponent in terms of a limit involving Fibonacci-like sequences.

Introduction

- A Lyapunov exponent is a characterizing quantity that appears in the study of measure-preserving dynamical systems.
- We focus on this quantity in the context of products of random matrices.

Definition (Lyapunov exponent)
Let $Y_1, Y_2, \dots$ be i.i.d. $\mathrm{GL}(d, \mathbb{R})$-valued random matrices and $S_n = \prod_{i=1}^n Y_i$. The Lyapunov exponent, λ, corresponding to the distribution of Y_1 is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\log \|S_n\|\right].$$

Theorem (Analogue to Law of Large Numbers) [Furstenberg and Kesten]
Let $Y_1, Y_2, \dots$ be i.i.d. $\mathrm{GL}(d, \mathbb{R})$-valued random matrices such that $\mathbb{E}[\log^+ \|Y_1\|] < \infty$ and $S_n = \prod_{i=1}^n Y_i$. Then,

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\log \|S_n\|$$

almost surely.

Theorem (Furstenberg's Theorem)
Let $Y_1, Y_2, \dots$ be i.i.d. $\mathrm{GL}(2, \mathbb{R})$-valued random matrices with common distribution μ and let λ be the associated Lyapunov exponent. Then there exists a random variable X with distribution ν such that

$$\int_{\mathbf{P}^2}\int_{\mathrm{GL}(2,\mathbb{R})} \phi(Y\cdot X)\,\mathrm{d}\mu(Y)\,\mathrm{d}\nu(X) = \int_{\mathbf{P}^2}\phi(X)\,\mathrm{d}\nu(X)$$

for any bounded, measurable ϕ. In particular, $Y_1 \cdot X$ is distributed the same as X, which we write as $Y_1 \cdot X \sim X$. Moreover, we have that $\lambda > 0$ almost surely, and

$$\lambda = \mathbb{E}[\log X].$$

Theorem (Analogue to Central Limit Theorem) [Le Page]
Let $Y_1, Y_2, \dots$ be i.i.d. $\mathrm{GL}(2, \mathbb{R})$-valued random matrices and $S_n = \prod_{i=1}^n Y_i$. For any $x \in \mathbf{P}^2$, there exists a Gaussian random variable Z with mean 0 and variance a^2 such that

$$\frac{1}{\sqrt{n}}\left(\log\|S_n\bar{x}\| - n\lambda\right) \to Z \text{ and } \frac{1}{\sqrt{n}}\left(\log\|S_n\| - n\lambda\right) \to Z.$$

Random Matrix Models

Bernoulli$\left(\frac{1}{2}\right)$

We consider the random matrix

$$Y_i = \begin{pmatrix} \epsilon_i & 1 \\ 1 & 0 \end{pmatrix},$$

where $\epsilon_i \sim$ Bernoulli $\left(\frac{1}{2}\right)$. The invariant distribution is of the form

$$X \sim \frac{1}{X} + \epsilon_i.$$

We show that

$$\begin{aligned}\lambda = \mathbb{E}\left[\log X\right] &= \frac{1}{6}\mathbb{E}\left[\log\left(2X+1\right)\right] \\ &= \frac{1}{14}\mathbb{E}\left[\log\left(3X+2\right)\left(X+2\right)\right] \\ &= \frac{1}{32}\mathbb{E}\left[\log\left(5X+3\right)\left(3X+1\right)\left(2X+3\right)\left(2X+1\right)\right] \\ &= \dots.\end{aligned}$$

We find that the coefficient pairs in these equalities take the form $(a_n^k,\ b_n^k)$, where n and k indicate a coefficient's position. Calculating the products of the pair sums,

$$\begin{aligned}
n=0 \quad & (2,1) \mapsto 3 \\
n=1 \quad & (3,2)\,(1,2) \mapsto 5\cdot 3 = 15 \\
n=2 \quad & (5,3)\,(3,1)\,(2,3)\,(2,1) \mapsto 8\cdot 4\cdot 5\cdot 3 = 480 \\
n=3 \quad & (8,5)\,(4,3)\,(5,2)\,(3,2)\,(3,5)\,(1,3)\,(2,1)\,(2,3) \mapsto 13\cdot 7\cdot 7\cdot 5\cdot 8\cdot 4\cdot 3\cdot 5 = 1528800 \\
n=4 \quad & (13,8)\,(7,4)\,(7,5)\,(5,3)\,(8,3)\,(4,1)\,(3,2)\,(5,2) \\
& (5,8)\,(3,4)\,(2,5)\,(2,3)\,(5,3)\,(3,1)\,(1,2)\,(3,2) \\
& \mapsto 59668697090000 \\
\cdots \quad & \cdots,
\end{aligned}$$

we can calculate our object of interest, c_n, with the following:

$$c_n = \prod_{k=1}^{2^n}\left(a_n^k + b_n^k\right) = c_{n-1}\prod_{k=1}^{2^{n-1}}\left(a_n^k + b_n^k\right) = \prod_{k=1}^{2^n} a_{n+1}^k.$$

Theorem (Obtaining the Lyapunov exponent)
Let

$$p_n = \frac{\log c_n}{(n+7)\,2^n} \le \mathbb{E}\left[\log X\right] \le \frac{\log c_n}{(n+4)\,2^n} = q_n.$$

Then,

$$p_n, q_n \to \mathbb{E}\left[\log X\right] = \lambda.$$

Results

n	q_n	p_n	$q_n - p_n$
1	0.2708	0.1693	0.1015
2	0.2572	0.1715	0.0857
3	0.2543	0.1780	0.0763
4	0.2478	0.1802	0.0676
5	0.2454	0.1841	0.0613
6	0.2419	0.1860	0.0559
7	0.2401	0.1886	0.0515
8	0.2378	0.1902	0.0476
9	0.2364	0.1920	0.0444
10	0.2348	0.1934	0.0414

We expect that $\lambda \approx 0.2165$.

Cauchy with a parameter

Our ξ-Cauchy model is based on the random matrix

$$Y = \begin{pmatrix} \xi\epsilon & -1 \\ 1 & 0 \end{pmatrix},$$

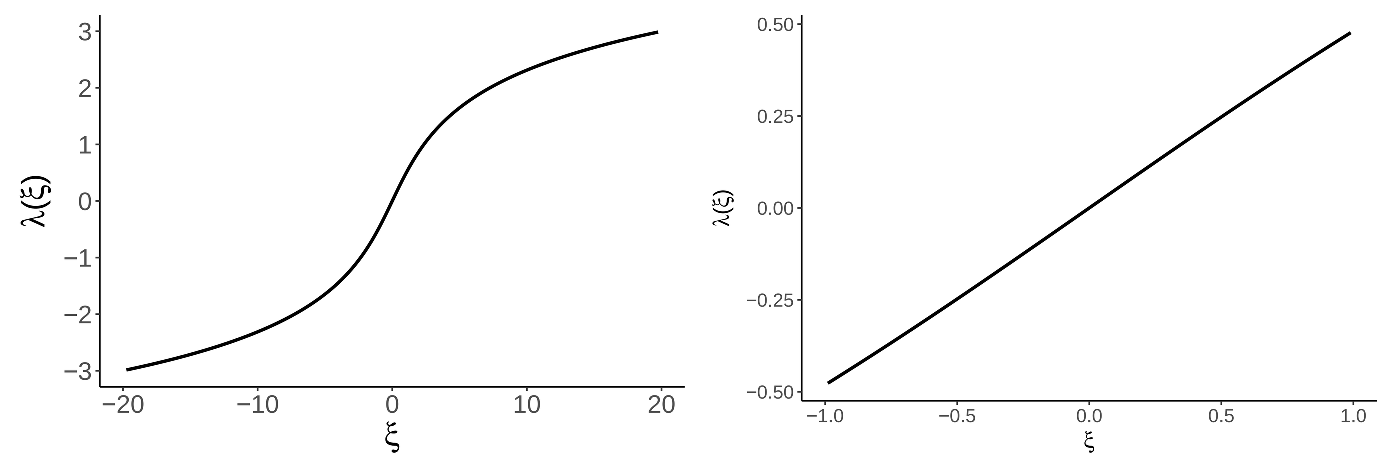
where $\epsilon \sim \mathrm{Cauchy}(0,1)$ and $\xi \in \mathbb{R}$. For this model, we can use the invariant distribution,

$$X \sim -\frac{1}{X} + \xi\epsilon,$$

to obtain an explicit formula for the Lyapunov exponent in terms of the parameter of interest, ξ.

Proposition In this model, the Lyapunov exponent is of the form

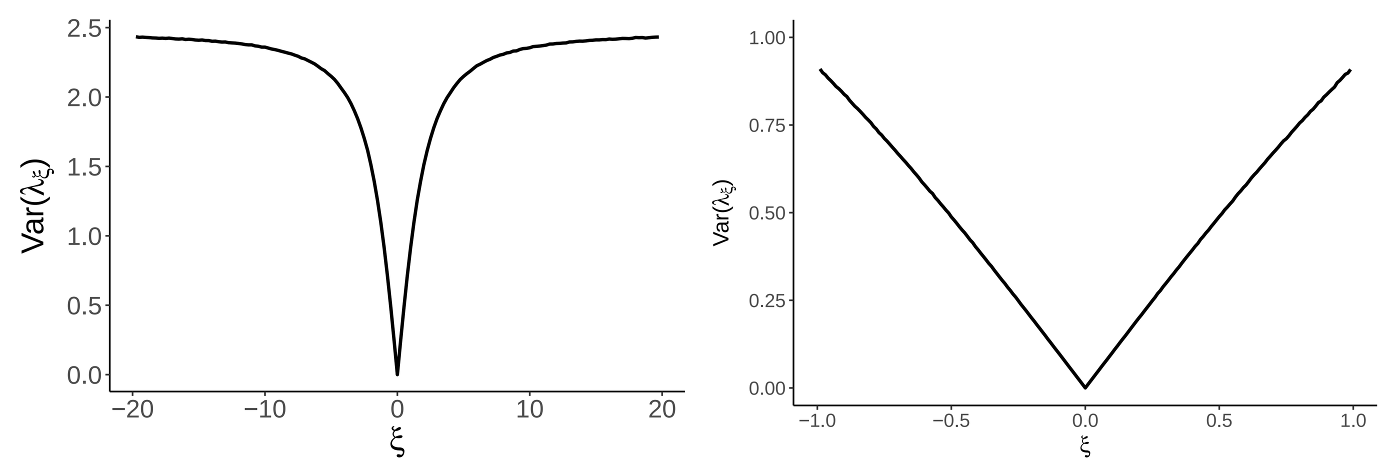
$$\lambda(\xi) = \log\left(\frac{\xi + \sqrt{\xi^2+4}}{2}\right).$$



(a) $\lambda(\xi)$ with $\xi \in [-20, 20]$ *(b)* $\lambda(\xi)$ with $\xi \in [-1, 1]$

Simulating the Variance of $\lambda(\xi)$

1. Choose an interval $[a, b]$ as the range of ξ. Divide this interval into sub-intervals of length k. Let ξ be of the form $a + jk$ for $j = 0, 1, \dots, \frac{b-a}{k} - 1$.
2. Choose a unit vector x.
3. For each ξ, simulate
$$L(\xi) = \frac{\sum_{i=1}^n \log\|S_i x\| - n\lambda(\xi)}{\sqrt{n}}$$
and store the result in a data vector of length $\frac{b-a}{k}$.
4. Repeat Step 3 an m number of times to obtain a $m \times \frac{b-a}{k}$ matrix, where the j^{th} column contains all of the $L(\xi)$ corresponding to ξ_j.
5. Calculate the variance of each column of the matrix.



(c) Variance with $\xi \in [-20, 20]$ *(d)* Variance with $\xi \in [-1, 1]$