Pricing the Black-Scholes European Call Option

We can derive the Black-Scholes formula on the premise that our prices follow a log-normal distribution.

Lemma 1. Assume there are no opportunities for arbitrage and the risk-free interest is r. Given a European call option with expiration t and strike K, let \( X_t \) be the time-t price of the underlying security, where \( X_0 = X_0e^{rt} \) for some \( Y_t = N(\mu, \sigma^2) \). Then the discounted price of the call option, denoted \( C \), is given by

\[
C = e^{-rt}E_q[\max(X_t - K, 0)] = X_tN(d_2) - Ke^{-rt}N(d_1)
\]

Proof. The price of a call option should be its discounted expected profit, that is, \( C = e^{-rt}E_q[\max(X_t - K, 0)] \), otherwise, there would be an opportunity for arbitrage, violating our initial assumption. The natural probability measure to study the price of a security is the time-t cash pricing measure \( Q \). Expected profit is thus a straightforward computation:

\[
E_q[\max(X_t - K, 0)] = \int_{0}^{\infty} \max(X_t - K, 0) q(y) dy = X_tN(d_2) - Ke^{-rt}N(d_1)
\]

and by discounting this formula, we obtain \( C = e^{-rt}E_q[\max(X_t - K, 0)] \).

Log-Normality of Prices

We use the Lindenberg-Feller Central Limit Theorem to prove the premise that our prices follow a log-normal distribution under the following assumptions.

Assumption 1. For each t, the random variable \( Y_t = \log S_t \) has mean \( \mu \) and finite variance.

Assumption 2. The differences \( Y_t - Y_s \) are independent for disjoint intervals \([s, t]\), for intervals of equal length, they are i.i.d.

Assumption 3. For every \( t > 0 \), \( x \in \mathbb{R} \), and \( \epsilon > 0 \), \( \epsilon \log(1 + \epsilon) \leq \epsilon x + x^2/2 \).

Lemma 2. Suppose \( f(0, \infty) = \infty \). Then \( \epsilon \log(1 + \epsilon) \leq \epsilon x + x^2/2 \).

Proof. Theorem 2. Under Assumptions 1, 2, and 3, for every \( t > 0 \), \( Y_t = \log S_t \) is a normal random variable with variance \( \sigma^2 \).

In addition to the Lindenberg-Feller central limit theorem, the proof of Theorem 2 makes use of the following lemma.

Lemma 3. Suppose \( f(0, \infty) = \infty \). Then \( \epsilon \log(1 + \epsilon) \leq \epsilon x + x^2/2 \).

Proof. Theorem 2. We first show that

\[
E(Y_t) = \mu + \sigma^2/2
\]

by Assumption 2. Via \( Y_{t_0} - Y_{t_0} = Y_{t_0} - Y_{t_0} + Y_{t_0} - Y_{t_0} \), thus establishing Equations 3. Now with \( X_{t_0} = Y_{t_0} \) and at each time \( t \), we obtain, by telescopic cancellation,

\[
S_t - Y_0 = \sum_{i=0}^{t-1} X_i
\]

where the dependence of \( X_{t_0} \) on \( t \) is suppressed for notational convenience. Clearly, \( X_{t_0} \) has mean 0.

By the assumption of stationarity, \( X_{t_0} \) is stochastically equal to \( Y_{t_0} \). Consequently, by Equation 3,

\[
E(Y_t) = \mu + \sigma^2/2
\]

implies \( \sum_{i=0}^{t} E(X_i^2) = t \sigma^2 \).

Finally, by the consequence of the assumption of stationarity noted above,

\[
\sum_{i=0}^{t} E(X_i^2) > \epsilon \epsilon \mathbb{E}[Y_{t_0}^2] > \epsilon \mathbb{E}[Y_{t_0}^2]
\]

whence the Lindenberg condition follows from Assumption 3.