Weak Anticipation in the General Discrete Market

Ayelet Amiran, Fabrice Baudoin, Skylyn Nicole Brock, Berend Coster, Ryan Craver, Ugonna Ezeaka, Phanuel Mariano, Mary Wishart

UCONN | UNIVERSITY OF CONNECTICUT

National Science Foundation

Abstract

Utility optimization, particularly in the stock market, has been studied by many people in multiple settings. We focus on the general discrete model with more specific discrete time applications which allow us to take into account multiple financial assets. Our research explores the value, in terms of utility, of having insider information about the market you are working in. Our main result provides the value of this information for a variety of utility functions in complete markets. We obtained these results using the martingale method and by working with weak anticipation.

1 Introduction

- Purpose: Investigate how to optimize a stock portfolio given weak information
- Utility functions analyzed in discrete time in the complete model
- Explored anticipation with use of martingale method
- Functions analyzed with respect to expected utility as opposed to expected wealth.

2 Utility Functions

- Utility functions denoted as $U(x)$
- Utility Function Conditions

  - Properties of Functions
    - strictly concave
    - strictly increasing
    - continuously differentiable
  - Effects of Conditions
    - risk aversion
    - satisfy the law of diminishing marginal utility
    - guarantee that an increase in wealth results in an increase in utility

- Looking at two different utility functions:

1. Log utility: $U(x) = \ln(x)$
2. Power utility: $U(x) = \frac{x^\gamma}{\gamma}$, for $0 < \gamma < 1$

3 Setup to our model

- Market with d financial assets
- Sample space $\Omega = \{\omega_1, \dots, \omega_M\}$ affecting the prices of the assets
- Let $\vec{S_n} \in \mathbb{R}^d$ and $V_n \in \mathbb{R}$ denote the asset prices of every asset and the value of wealth at time n respectively, for $n \in \{0, 1, \dots, N\}$
- $V_0 = v$
- Define r to be the risk-free interest rate
- Define $\mathcal{M}$ as the set of equivalent probability measures under which our discounted asset prices are martingales

4 Weak Anticipation

- Suppose we are in a complete market:
  - Every financial derivative can be replicated
  - $\mathcal{M} = \{\tilde{\mathbb{P}}\}$
- Suppose we have some weak anticipation (weak information) regarding the prices of asset prices at our final time period, denoted by ν

Definition 4.1. *The probability measure $\mathbb{P}^\nu$ defined by*

$$\mathbb{P}^\nu(B) := \sum_{\vec{x} \in \mathcal{A}} \tilde{\mathbb{P}}(B|\vec{S_N} = \vec{x})\nu(\vec{S_N} = \vec{x})$$

is the minimal probability measure associated with the weak information ν, where $\tilde{\mathbb{P}} \in \mathcal{M}$ is the only equivalent martingale measure.

Proposition 4.2. *Let ϕ be a convex function. Then*

$$\min_{\mathbb{Q} \in \mathcal{E}^\nu} \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\right)\right] = \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\right)\right]$$

where $\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}$ represents the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\tilde{\mathbb{P}}$.

Proof This proposition follows from the conditional Jensen inequality for convex functions, the definition of conditional probability, and some computations.

5 Value of Weak Information in a Complete Market

Definition 5.1. *The **financial value of weak information** is the lowest increase in utility that can be gained from anticipation. We write*

$$u(v,\nu) = \min_{\mathbb{Q} \in \mathcal{E}^\nu} \max_{\psi \in \Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)].$$

Theorem 5.2. *The financial value of weak information in a complete market is*

$$u(v,\nu) = \max_{\psi \in \Psi^v} \mathbb{E}^\nu[U(V_N)] = \mathbb{E}^\nu\left[U\left(I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right],$$

where $\lambda(v)$ is determined by

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v,$$

where $\tilde{\mathbb{P}} \in \mathcal{M}$ is the unique probability measure under which the prices are martingales and $I(z) = (U')^{-1}(z)$.

Proof This proof follows from setting up a Lagrangian, the use of the convex conjugate of U, the martingale method, and Proposition 4.2.

5.1 Application of Weak Information to Utility Functions

We can apply the Value of Weak Information to our Log and Power utilities to give the following results.

Log Utility

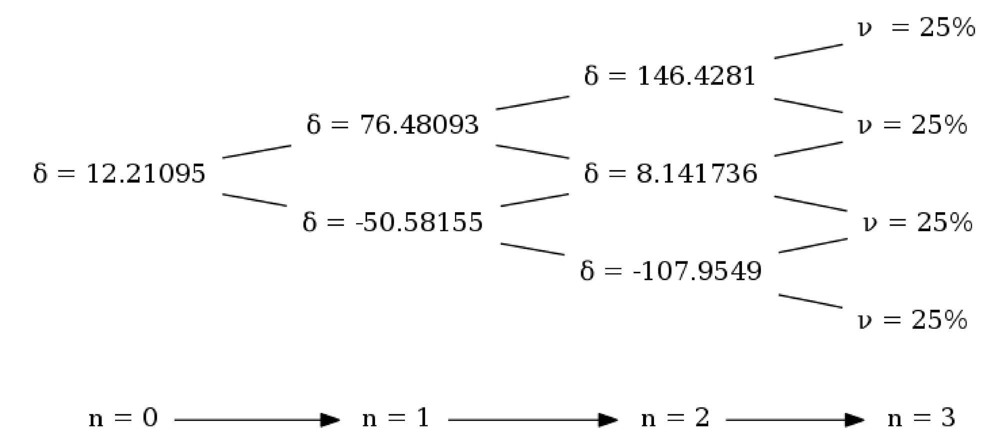
$$u(v,\nu) = \ln\left(v(1+r)^N\right) + \mathbb{E}^\nu\left[\ln\left(\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\right)\right]$$

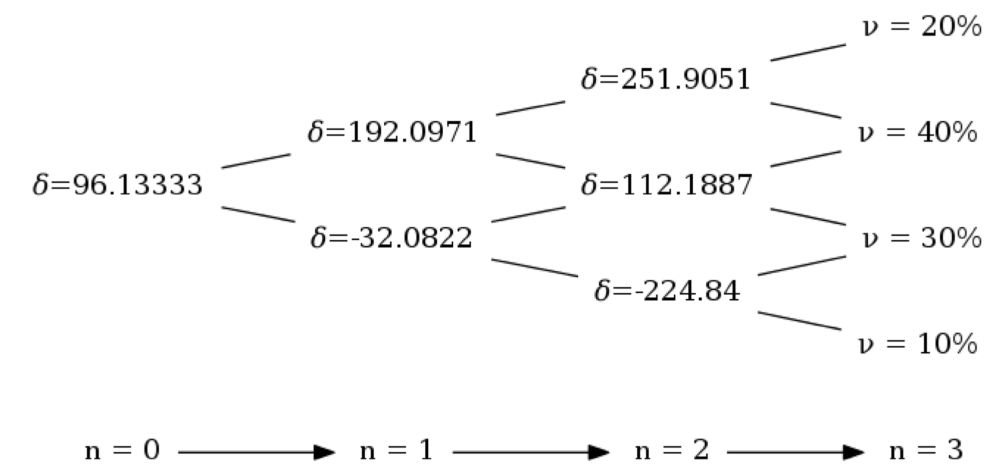
Power Utility

$$u(v,\nu) = \frac{v^\gamma(1+r)^{N\gamma}}{\gamma\left(\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)^{\frac{1}{\gamma-1}}\right]\right)^{\gamma-1}} \cdot \mathbb{E}^\nu\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)^{\frac{\gamma}{\gamma-1}}\right]$$

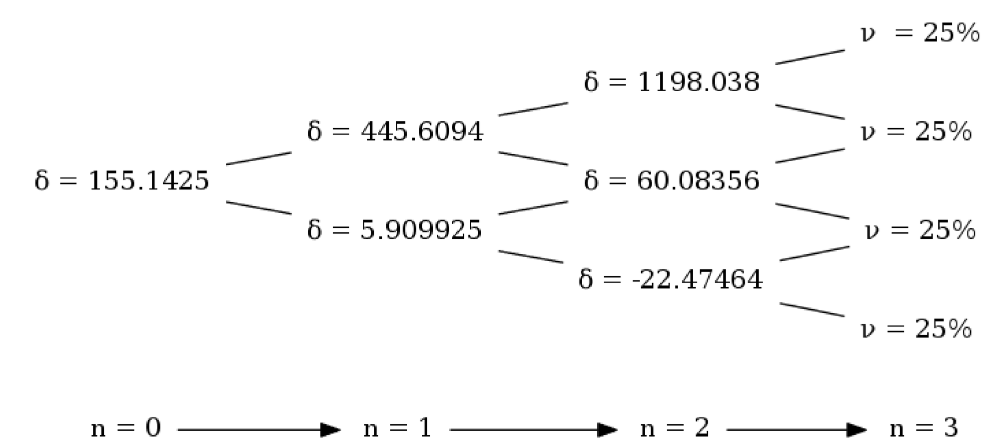
6 Binomial Model

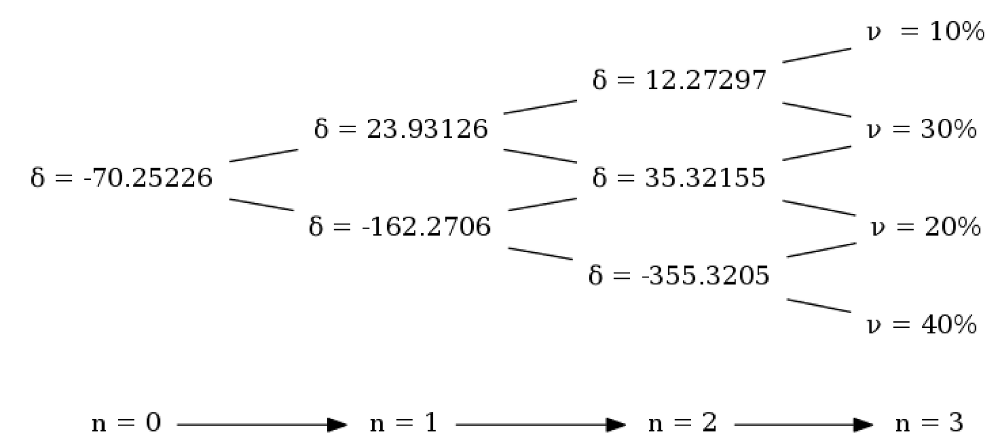
- One risk-free asset with payoff $1 + r$
- One risky asset that pays either $S_0(1+h)$ or $S_0(1-k)$
- Only two possible outcomes of the market
- δ is the optimal amount of stock one should buy at a time n





Two 3-period binomial trees using the log utility, one with uniform distribution of the endpoints and one without, where the parameters are $r = .032$, $h = .09$, $k = .019$, $v = 200.0$, and $s = 20.0$.





Two 3-period binomial trees using the same parameters and distributions but instead using power utility. The value of $\gamma = .5$

Research supervised by Dr. Fabrice Baudoin and supported by the Department of Mathematics at the University of Connecticut and the National Science Foundation. Bibliography available on request.