Gradients on Higher Dimensional Sierpiński Gaskets

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What is the Sierpiński Gasket?

Definition

Let \( \{p_i\}_{i=0}^{N-1} \) be the vertices of a standard \( N-1 \)-simplex, embedded in \( \mathbb{R}^{N-1} \).

- We define the family of maps, \( F_i(x) = \frac{1}{2}(x - p_i) + p_i \)
- Note: We call \( \{F_i\}_i \) is an iterated function system (IFS)
- \( SG_N \) is the unique non-empty compact set \( K \) such that

\[
K = \bigcup_{i=0}^{N-1} F_i K
\]
What is the Sierpiński Gasket?

Figure 1: $SG_3$ first level

Figure 2: $SG_3$ second level

Figure 3: $SG_4$ first level

Figure 4: $SG_4$ second level
Infinite Words

**Definition**

Let $N \in \mathbb{N}$, and $S_N = \{0, 1, \ldots, N - 1\}$.

We define $\Omega_N = S_N^\mathbb{N}$ to be the space of infinite words over $S_N$, and we will call $\omega \in \Omega_N$ an infinite word in $\Omega_N$. Example on $SG_4$, we have that $S_N = \{0, 1, 2, 3\}$:

$$\omega_1 = 00121120030201331210210\ldots$$
Truncated Words

Definition

Let $N \in \mathbb{N}$, and let $\omega \in \Omega_N$ such that $\omega = \omega_1\omega_2 \cdots$ with each $\omega_i \in S_N$.

We will call $[\omega]_n \in S^m_N$ a **truncated word**, defined by $[\omega]_n = \omega_1\omega_2 \cdots \omega_n$. Example using the previous word, truncated at $n = 10$:

$$[\omega_1]_{10} = 0012112003$$
Using words as addresses

We can identify an $n$-level cell $C$ using the word $[\omega]_n$, where each letter $k$ corresponds to the mapping $F_k$, and $F[\omega]_n K = C$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures}
\caption{$SG_3$ first level \hspace{1cm} $SG_3$ second level}
\end{figure}

Note that then infinite words correspond to points on $SG_N$. 
Graph Approximations

We consider $SG$ as a graph-motivated fractal: approximated by a sequence of simple graphs $\Gamma_n$, such that $\lim_{n \to \infty} \Gamma_n = SG$.

![Graph Approximations](image)

**Figure 7:** $\Gamma_1$ and $\Gamma_2$
We define real-valued functions on $SG_N$. Below we approximate a function on $SG_3$ by only considering the words on $\Gamma_1, \Gamma_2, \Gamma_3$.

Figure 8: Approximations of a function
A **probability measure** on a self-similar structure $K$ is a measure $\mu$ such that $\mu(K) = 1$. A **self-similar probability measure** on $K$ is one that assigns measure weights $\{\mu_i\}$ to the self-similar components, or cells, of $K$. For each cell of $K$, the distribution of the measure is the same as that of the whole object.

We can see a self-similar measure for $SG$ below:

The **standard** self-similar measure on $SG_N$ is one where $\mu_i = \frac{1}{N}$ for all $i$. 
Defining the Laplacian

**Definition**

Let \( u_m : V(\Gamma_m) \to \mathbb{R} \). We define the graph Laplacian

\[
\Delta_m u(x) = \sum_{y \in N(x)} [u(y) - u(x)]
\]

and the Laplacian

\[
\Delta_\mu u(x) = \lim_{m \to \infty} \left( \frac{N + 2}{N} \right)^m \mu_{m,x}^{-1} \Delta_m u(x)
\]

on \( SG_N \), where \( N(x) \) refers to the neighbors of \( x \) at the level \( m \) and \( \mu_{m,x} \) is a renormalization factor centered at \( x \).

Consider the unit interval with \( u(x) = x^2 \). Here, we plainly have \( \Delta_m u(0) = \left( \frac{1}{2^m} \right)^2 = \frac{1}{4^m} \).
Harmonic Functions and Harmonic Extension

Definition

**Harmonic functions** are continuous functions $h$ for which $\Delta_\mu h = 0$.

On the unit interval, these are linear functions.

- Harmonic functions are uniquely determined by their boundary values.
- Let $v$ be the boundary values of harmonic function $h$ on cell $C$, $v^*$ the boundary values of $h$ on cell $F_iC$. Then, there exists an invertible matrix $A_i$ such that $v^* = A_i v$.
Definition

The space of harmonic functions on $SG_N$ is denoted $\mathcal{H}$, and is $N$-dimensional. We obtain $N - 1$-dimensional $\tilde{\mathcal{H}}$ as the orthogonal complement of the constant vector $(1, \ldots, 1)^T$ with respect to the Euclidean inner product.
More on Harmonic Extensions

Let $P: \mathcal{H} \rightarrow \widetilde{\mathcal{H}}$ be the orthogonal projection along the constant subspace. Given a harmonic extension matrix $A_i$, we can define $\widetilde{A}_i = PA_iP$ as an operator on $\widetilde{\mathcal{H}}$.

As a small, but very useful lemma for our project, we found that for each $i$ in $SG_N$:

- $A_i$ has eigenvalues $\{1, \frac{1}{N+2}, \frac{N}{N+2}\}$,
- $\widetilde{A}_i$ has eigenvalues $\{\frac{1}{N+2}, \frac{N}{N+2}\}$. 
Let $h$ be the harmonic function that agrees with $u$ on the boundary of $F_{[\omega]} I$. We have

\[
\begin{pmatrix}
    h(0)
    \\
    h(1)
\end{pmatrix}
= A_{\omega_1}^{-1} \ldots A_{\omega_n}^{-1}
\begin{pmatrix}
    u(F_{[\omega]} 0)
    \\
    u(F_{[\omega]} 1)
\end{pmatrix}
= h
\]
Secants on the Sierpiński Gasket

We generalize this to $SG_N$:

$$h = A^{-1}_{[\omega]_n} u(F_{[\omega]_n} V_0)$$

This is a harmonic function which agrees with $u$ on the boundary of a specified level-$n$ cell.

We can define a ”tangent” at $\omega$ by taking the limit of these secants as $n \to \infty$, if such a limit exists.
Definition

Let \( u : SG_N \to \mathbb{R} \). We define the approximants:

\[
\nabla_n u(\omega) = \tilde{A}_{[\omega]}^{-1} P u(F_{[\omega]} V_0).
\]

Then, if the following limit exists, we define the gradient:

\[
\nabla u(\omega) = \lim_{n \to \infty} \nabla_n u(\omega)
\]
Teplyaev’s Theorem and Corollary

**Theorem (Teplyaev, 1998)**

Suppose \( f \in \text{Dom}\Delta_\mu \). Then, \( \nabla f(\omega) \) exists for every \( \omega \in \Omega \) such that

\[
\sum_{n \geq 1} r[\omega]_n \mu[\omega]_n \left\| \tilde{A}^{-1}[\omega]_n \right\| < \infty
\]

**Corollary (Teplyaev, 1998)**

Suppose that \( f \in \text{Dom}\Delta_\mu \). Then, \( \nabla f(\omega) \) exists for all \( \omega \in \Omega \) if

\[
r_j \mu_j \| \tilde{A}^{-1}_j \| < 1
\]

For \( j = 1, \ldots, N \). Moreover, in this case, \( \nabla f(\omega) \) is continuous in \( \Omega \).
Our Motivating Questions

We considered the following questions when conducting our research:

- How can we apply Teplyaev’s corollary to $SG_N$?
- Can we generalize Teplyaev’s results on the Sierpiński Gasket to $SG_N$?
- Is continuity of the Laplacian enough to guarantee the existence of the gradient on $SG_N$ for $N > 3$?
Applying the Corollary to the Sierpiński Gasket

We would like to apply Teplyaev’s corollary on the Sierpiński Gasket with the self-similar measure above. The resistance for the Sierpiński Gasket is $\frac{3}{5}$. For each harmonic extension matrix $\tilde{A}_j$, we have that $\|\tilde{A}_j^{-1}\| = 5$. Thus, $\nabla f(\omega)$ exists for all $\omega \in \Omega$ if

$$\frac{3}{5} \mu_j 5 = 3 \mu_j < 1 \Rightarrow \mu_j < \frac{1}{3}$$

for all $0 \leq j \leq 2$, but this doesn’t work because that would imply $\mu_0 + \mu_1 + \mu_2 < 1$, so the corollary cannot be used.
We run into similar issues when trying to apply the corollary to $SG_N$. We will then need to modify the way we approach these fractals in order to use Teplyaev’s corollary.
Trying Different Self-Similar Measures

Fractal Gradients
Karuna Sangam

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References

Trying Different Self-Similar Measures

\[
\begin{array}{c}
p_0 \\
\mu_{00} \\
\mu_{01} \quad \mu_{02} \\
\mu_{10} \quad \mu_{11} \quad \mu_{12} \\
\mu_{20} \quad \mu_{21} \quad \mu_{22} \\
p_1 \\
p_2
\end{array}
\]
We can use different self-similar measures on the Sierpiński Gasket, like the one previously shown. The resistance in this case is equal to \( \left( \frac{3}{5} \right)^2 \).

In addition, the harmonic extension matrices are of the form

\[
\left\| \tilde{A}_{ij}^{-1} \right\| = \left\| \left( \tilde{A}_j \tilde{A}_i \right)^{-1} \right\| = \begin{cases} 25 & i = j \\ \frac{25}{9} \sqrt{17 + 4 \sqrt{13}} & i \neq j \end{cases}
\]

Thus \( \nabla f(\omega) \) exists for all \( \omega \in \Omega \) if

\[
\left( \frac{3}{5} \right)^2 \mu_{ij} \left\| \tilde{A}_{ij}^{-1} \right\| < \begin{cases} 9 \mu_{ij} & i = j \\ \mu_{ij} \sqrt{17 + 4 \sqrt{13}} & i \neq j \end{cases}
\]

\[
\Rightarrow \mu_{ij} < \begin{cases} \frac{1}{9} & i = j \\ \frac{1}{\sqrt{17 + 4 \sqrt{13}}} & i \neq j \end{cases}
\]
Trying Different Self-Similar Measures
The following is a generalization of a theorem by Teplyaev regarding the existence of the gradient on a specific set of words given the standard self-similar measure.

**Theorem**

Let $\mu$ be the standard measure on $SG_N$, let $u: SG_N \to \mathbb{R}$ be a function, and suppose $\Delta_\mu u$ is continuous. Then, $\nabla u(\omega)$ is defined at every $\omega \in \Omega$ such that

$$\liminf_{n \to \infty} \frac{C_N(\omega, n)}{\log n} \geq \gamma$$

Where $\gamma > 0$ is a certain constant.
On the previous slide, we mentioned the change counting function $C_N(\omega, n)$.

Given some $N \in \mathbb{N}$ and a word $\omega = \omega_1\omega_2 \ldots$ in $\Omega_N$, the counting function $C_N(\omega, n)$ counts the number of instances in $[\omega]_n$ where $N - 1$ distinct letters appear consecutively.

For example, if we take $\omega = 011322303003221213211 \ldots$, then we get that $C_4(\omega, 21) = 6.$
A Counterexample Function

Suppose $u$ is in the domain of $\Delta_\mu$. Is this enough to guarantee that $\nabla u(\omega)$ exists for every $\omega \in \Omega$?

To show that it is not, we constructed a counterexample function, $\Phi$.

- $\Phi$ is in the domain of $\Delta_\mu$, and so, by definition, $\Delta_\mu \Phi$ is continuous on $SG_4$.
- The gradient on a particular edge of $SG_4$ diverges.
Conjecture for a Hölder Continuous Laplacian

We are close to completing our proof of the following result.

**Theorem**

Let \( u : SG_N \to \mathbb{R} \), and let \( \mu \) be the standard measure on \( SG_N \). If \( \Delta_\mu u \) is Hölder continuous, then \( \nabla u(\omega) \) exists for all \( \omega \in \Omega \).
In addition to proving the previous conjecture, we are interested in exploring other directions with this project.

In particular, we want to study how using non-self-similar measures, specifically the **Kusuoka measure**, affects the conditions necessary to guarantee the existence and continuity of the gradient.

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Thank you for listening!