Gradients on Higher Dimensional Sierpiński Gaskets

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Outline

What is the Sierpiński Gasket?

Definition

Let $\{p_i\}_{i=0}^{N-1}$ be the vertices of a standard $N-1$-simplex, embedded in $\mathbb{R}^{N-1}$.

- We define the family of maps, $F_i(x) = \frac{1}{2}(x - p_i) + p_i$
- Note: We call $\{F_i\}_i$ is an **iterated function system** (IFS)
- SG_N is the unique non-empty compact set K such that

$$K = \bigcup_{i=0}^{N-1} F_i K$$

What is the Sierpiński Gasket?

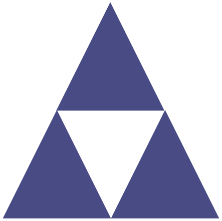


Figure 1: SG_3 first level

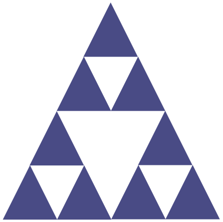


Figure 2: SG_3 second level

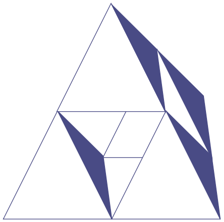


Figure 3: SG_4 first level

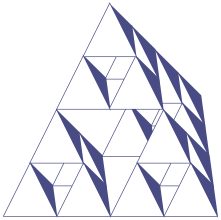


Figure 4: SG_4 second level

Infinite Words

Definition

Let $N \in \mathbb{N}$, and $S_N = \{0, 1, \ldots, N-1\}$.

We define $\Omega_N = S_N^{\mathbb{N}}$ to be the **space of infinite words** over S_N, and we will call $\omega \in \Omega_N$ an **infinite word** in Ω_N.
Example on SG_4, we have that $S_N = \{0, 1, 2, 3\}$:

$$\omega_1 = 001211200302013312102 10...$$

Truncated Words

Definition

Let $N \in \mathbb{N}$, and let $\omega \in \Omega_N$ such that $\omega = \omega_1\omega_2\cdots$ with each $\omega_i \in S_N$.

We will call $[\omega]_n \in S_N^n$ a **truncated word**, defined by $[\omega]_n = \omega_1\omega_2\cdots\omega_n$. Example using the previous word, truncated at $n = 10$:

$$[\omega_1]_{10} = 0012112003$$

Using words as addresses

We can identify an n-level cell C using the word $[\omega]_n$, where each letter k corresponds to the mapping F_k, and $F_{[\omega]_n} K = C$.

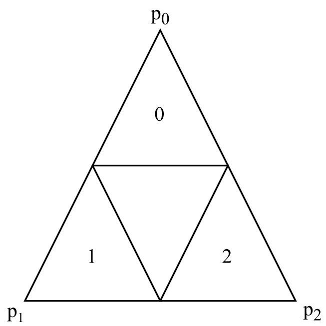


Figure 5: SG_3 first level

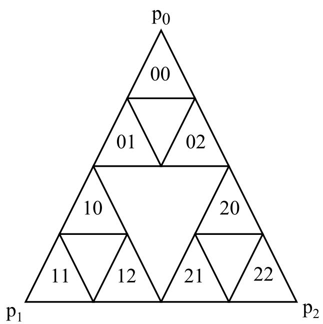


Figure 6: SG_3 second level

Note that then infinite words correspond to points on SG_N.

Graph Approximations

We consider SG as a graph-motivated fractal: approximated by a sequence of simple graphs Γ_n, such that $\lim_{n\to\infty} \Gamma_n = SG$.

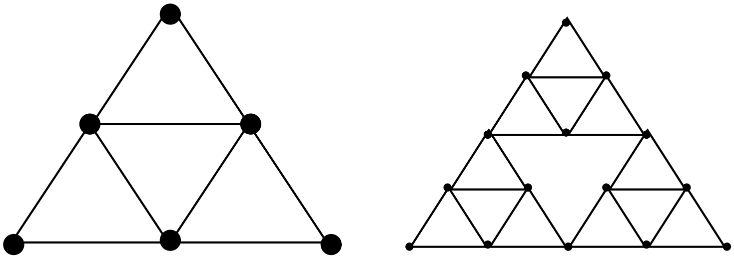


Figure 7: Γ_1 and Γ_2

Defining functions on Fractals

We define real-valued functions on SG_N. Below we approximate a function on SG_3 by only considering the words on $\Gamma_1, \Gamma_2, \Gamma_3$.

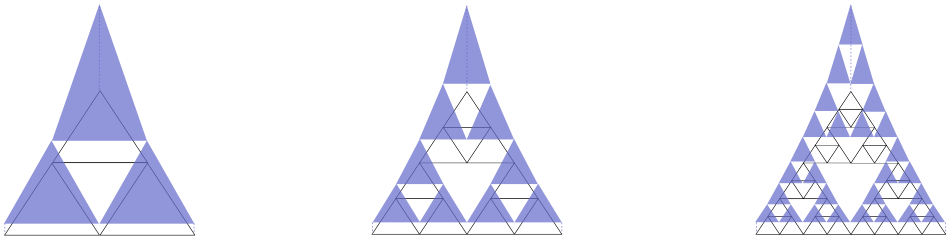
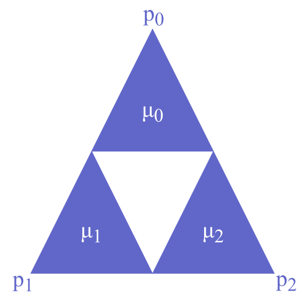


Figure 8: Approximations of a function

Self-Similar Measures

A **probability measure** on a self-similar structure K is a measure μ such that $\mu(K) = 1$. A **self-similar probability measure** on K is one that assigns measure weights $\{\mu_i\}$ to the self-similar components, or cells, of K. For each cell of K, the distribution of the measure is the same as that of the whole object.
We can see a self-similar measure for SG below:



The **standard** self-similar measure on SG_N is one where $\mu_i = \frac{1}{N}$ for all i.

Defining the Laplacian

Definition

Let $u_m : V(\Gamma_m) \to \mathbb{R}$. We define the **graph Laplacian**

$$\Delta_m u(x) = \sum_{y \in N(x)} [u(y) - u(x)]$$

and the **Laplacian**

$$\Delta_\mu u(x) = \lim_{m \to \infty} \left(\frac{N+2}{N} \right)^m \mu_{m,x}^{-1} \Delta_m u(x)$$

on SG_N, where $N(x)$ refers to the neighbors of x at the level m and $\mu_{m,x}$ is a renormalization factor centered at x.

Consider the unit interval with $u(x) = x^2$. Here, we plainly have $\Delta_m u(0) = \left(\frac{1}{2^m}\right)^2 = \frac{1}{4^m}$.

Harmonic Functions and Harmonic Extension

Definition

Harmonic functions are continuous functions h for which $\Delta_\mu h = 0$.

On the unit interval, these are linear functions.

- Harmonic functions are uniquely determined by their boundary values.
- Let v be the boundary values of harmonic function h on cell C, v^* the boundary values of h on cell F_iC. Then, there exists an invertible matrix A_i such that $v^* = A_i v$

Space of Harmonic Functions

Definition

The space of harmonic functions on SG_N is denoted $\mathcal{H}$, and is N-dimensional. We obtain $N-1$-dimensional $\widetilde{\mathcal{H}}$ as the orthogonal complement of the constant vector $(1,\ldots,1)^T$ with respect to the Euclidean inner product.

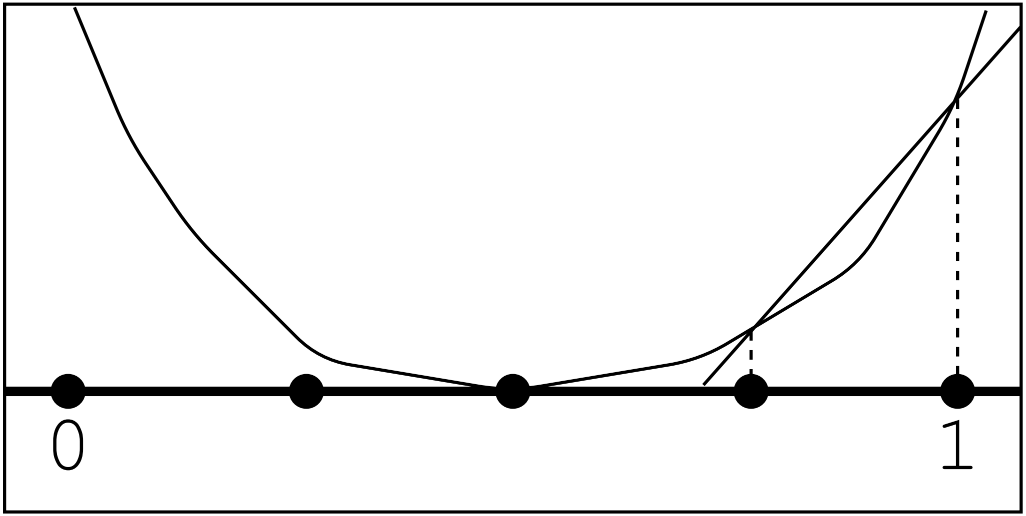
More on Harmonic Extensions

Let $P\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ be the orthogonal projection along the constant subspace. Given a harmonic extension matrix A_i, we can define $\widetilde{A}_i = PA_iP$ as an operator on $\widetilde{H}$.

As a small, but very useful lemma for our project, we found that for each i in SG_N:

- A_i has eigenvalues $\{1, \frac{1}{N+2}, \frac{N}{N+2}\}$,
- $\widetilde{A}_i$ has eigenvalues $\{\frac{1}{N+2}, \frac{N}{N+2}\}$.

Secants on the Unit Interval



Let h be the harmonic function that agrees with u on the boundary of $F_{[\omega]_n} I$. We have

$$\begin{pmatrix} h(0) \\ h(1) \end{pmatrix} = A_{\omega_1}^{-1} \dots A_{\omega_n}^{-1} \begin{pmatrix} u(F_{[\omega]_n} 0) \\ u(F_{[\omega]_n} 1) \end{pmatrix} = h$$

Secants on the Sierpiński Gasket

We generalize this to SG_N:

$$h = A_{[\omega]_n}^{-1} u(F_{[\omega]_n} V_0)$$

This is a harmonic function which agrees with u on the boundary of a specified level-n cell.

We can define a "tangent" at ω by taking the limit of these secants as $n \to \infty$, if such a limit exists.

Teplyaev's Gradient

Definition

Let $u : SG_N \to \mathbb{R}$. We define the **approximants**:

$$\nabla_n u(\omega) = \widetilde{A}_{[\omega]_n}^{-1} P u(F_{[\omega]_n} V_0).$$

Then, if the following limit exists, we define the **gradient**:

$$\nabla u(\omega) = \lim_{n\to\infty} \nabla_n u(\omega)$$

Teplyaev's Theorem and Corollary

Theorem (Teplyaev, 1998)

Suppose $f \in Dom\Delta_\mu$. Then, $\nabla f(\omega)$ exists for every $\omega \in \Omega$ such that

$$\sum_{n \geq 1} r_{[\omega]_n} \mu_{[\omega]_n} \left\| \widetilde{A}^{-1}_{[\omega]_n} \right\| < \infty$$

Corollary (Teplyaev, 1998)

Suppose that $f \in Dom\Delta_\mu$. Then, $\nabla f(\omega)$ exists for all $\omega \in \Omega$ if

$$r_j \mu_j \|\widetilde{A}_j^{-1}\| < 1$$

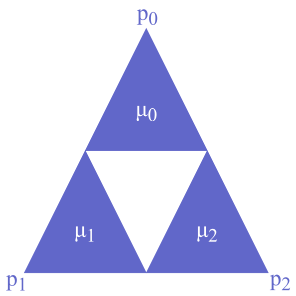
For $j = 1, \ldots, N$. Moreover, in this case, $\nabla f(\omega)$ is continuous in Ω.

Our Motivating Questions

We considered the following questions when conducting our research:

- How can we apply Teplyaev's corollary to SG_N?
- Can we generalize Teplyaev's results on the Sierpiński Gasket to SG_N?
- Is continuity of the Laplacian enough to guarantee the existence of the gradient on SG_N for $N > 3$?

Applying the Corollary to the Sierpiński Gasket



We would like to apply Teplyaev's corollary on the Sierpiński Gasket with the self-similar measure above. The resistance for the Sierpiński Gasket is $\frac{3}{5}$. For each harmonic extension matrix $\tilde{A}_j$, we have that $\left\| \tilde{A}_j^{-1} \right\| = 5$.
Thus, $\nabla f(\omega)$ exists for all $\omega \in \Omega$ if

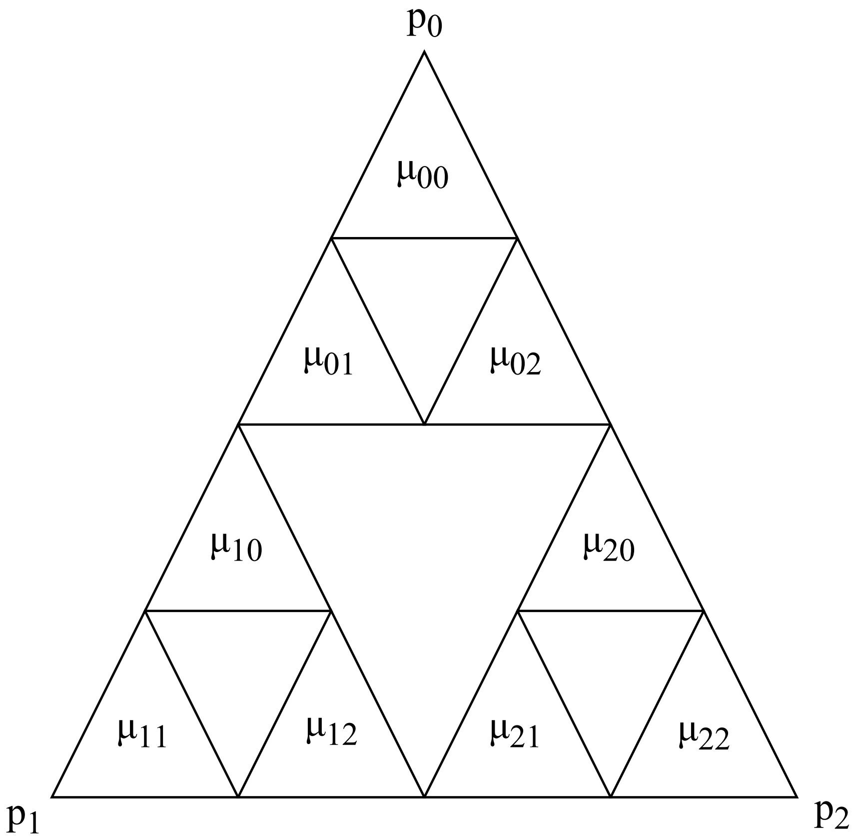
$$\frac{3}{5}\mu_j 5 = 3\mu_j < 1 \Rightarrow \mu_j < \frac{1}{3}$$

for all $0 \leq j \leq 2$, but this doesn't work because that would imply $\mu_0 + \mu_1 + \mu_2 < 1$, so the corollary cannot be used.

Applying the Corollary to the Sierpinski Gasket

We run into similar issues when trying to apply the corollary to SG_N. We will then need to modify the way we approach these fractals in order to use Teplyaev's corollary.

Trying Different Self-Similar Measures



Trying Different Self-Similar Measures

- We can use different self-similar measures on the Sierpiński Gasket, like the one previously shown.
- The resistance in this case is equal to $\left(\frac{3}{5}\right)^2$.
- In addition, the harmonic extension matrices are of the form

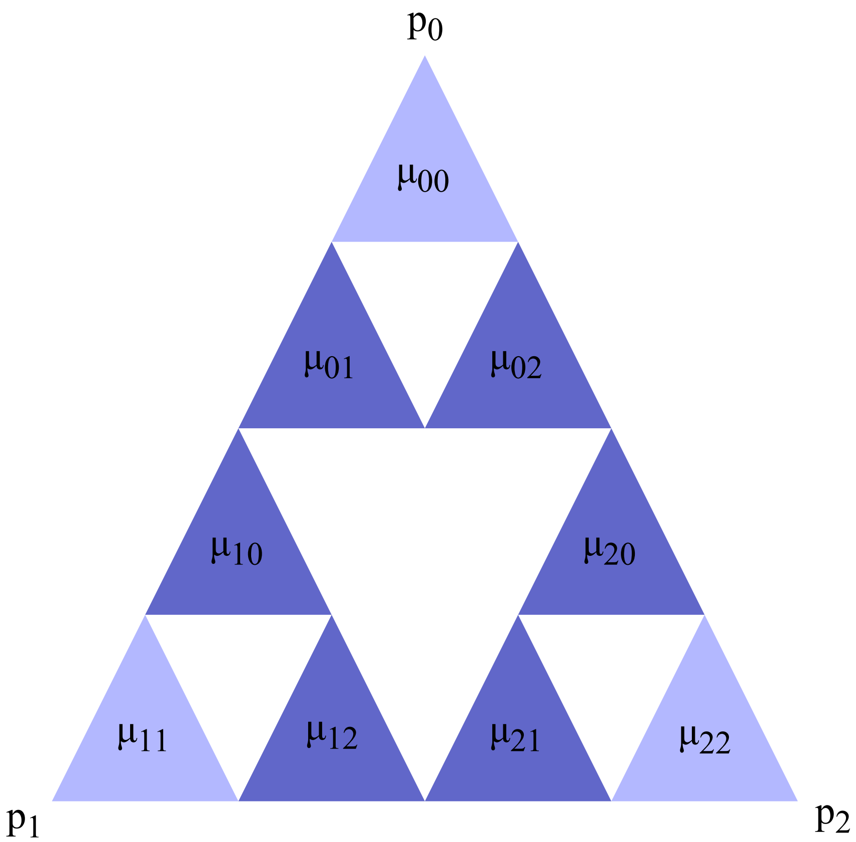
$$\left\| \tilde{A_{ij}}^{-1} \right\| = \left\| \left(\tilde{A}_j \tilde{A}_i \right)^{-1} \right\| = \begin{cases} 25 & i = j \\ \frac{25}{9}\sqrt{17 + 4\sqrt{13}} & i \neq j \end{cases}.$$

- Thus $\nabla f(\omega)$ exists for all $\omega \in \Omega$ if

$$\left(\tfrac{3}{5}\right)^2 \mu_{ij} \left\| \tilde{A_{ij}}^{-1} \right\| < \begin{cases} 9\mu_{ij} & i = j \\ \mu_{ij}\sqrt{17 + 4\sqrt{13}} & i \neq j \end{cases}$$

$$\Rightarrow \mu_{ij} < \begin{cases} \frac{1}{9} & i = j \\ \frac{1}{\sqrt{17+4\sqrt{13}}} & i \neq j \end{cases}.$$

Trying Different Self-Similar Measures



Generalizing a Theorem of Teplyaev

The following is a generalization of a theorem by Teplyaev regarding the existence of the gradient on a specific set of words given the standard self-similar measure.

Theorem

Let μ be the standard measure on SG_N, let $u\colon SG_N \to \mathbb{R}$ be a function, and suppose $\Delta_\mu u$ is continuous. Then, $\nabla u(\omega)$ is defined at every $\omega \in \Omega$ such that

$$\liminf_{n\to\infty} \frac{C_N(\omega, n)}{\log n} \geq \gamma$$

Where $\gamma > 0$ is a certain constant.

The Counting Function

On the previous slide, we mentioned the **change counting function** $C_N(\omega, n)$.

Given some $N \in \mathbb{N}$ and a word $\omega = \omega_1\omega_2\dots$ in Ω_N, the counting function $C_N(\omega, n)$ counts the number of instances in $[\omega]_n$ where $N-1$ distinct letters appear consecutively.

For example, if we take $\omega = 011322303003221213211\dots$, then we get that $C_4(\omega, 21) = 6$.

A Counterexample Function

Suppose u is in the domain of Δ_μ. Is this enough to guarantee that $\nabla u(\omega)$ exists for every $\omega \in \Omega$?

To show that it is not, we constructed a counterexample function, Φ.

- Φ is in the domain of Δ_μ, and so, by definition, $\Delta_\mu \Phi$ is continuous on SG_4.
- The gradient on a particular edge of SG_4 diverges.

Conjecture for a Hölder Continuous Laplacian

We are close to completing our proof of the following result.

Theorem

Let $u : SG_N \to \mathbb{R}$, and let μ be the standard measure on SG_N. If $\Delta_\mu u$ is Hölder continuous, then $\nabla u(\omega)$ exists for all $\omega \in \Omega$.

Future Research

In addition to proving the previous conjecture, we are interested in exploring other directions with this project.

In particular, we want to study how using non-self-similar measures, specifically the **Kusuoka measure**, affects the conditions necessary to guarantee the existence and continuity of the gradient.

References

Alexander Teplyaev. Gradients on Fractals. Journal of Functional Analysis, Elsevier, 1998. McMaster University.

Robert Strichartz. Differential Equations on Fractals. Princeton University Press, Princeton, NJ, 08540, 2006. Cornell University.

Acknowledgements

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Thank you for listening!