Wave Propagation through a Fractal Medium

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Introduction

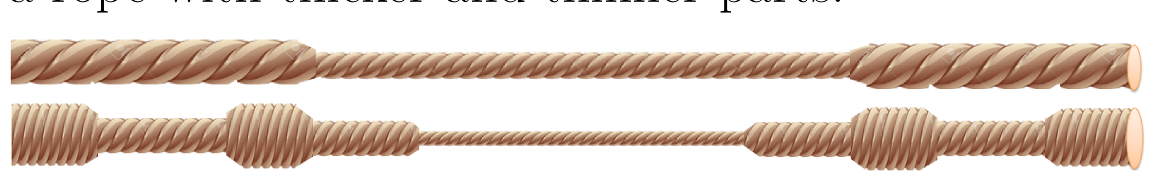
We consider the wave equation on the unit interval with fractal measure, and use two numerical models to study wave speed and propagation distance. The first approach uses a Fourier series of eigenfunctions of the fractal Laplacian, while the second uses a Markov chain to model the transmission and reflection of classical waves on an approximation of the fractal. These models have complementary advantages and limitations, and we conjecture that they approximate the same fractal wave.

Fractal Mass on the Unit Interval

We construct a family of fractal measures depending on a parameter $p \in (0, 1/2)$. Let $q = 1 - p$.

- Divide the unit interval into three subintervals with lengths $r_1 = \frac{p}{1+p}$, $r_2 = \frac{q}{1+p}$ and $r_3 = r_1$.
- Distribute mass in proportions $m_1 = \frac{q}{1+q}$, $m_2 = \frac{p}{1+q}$, $m_3 = m_1$.
- Repeat this procedure inside each interval, iterating infinitely many times.
- The limiting fractal measure is called μ.

Physically, one can think of waves in the approximations to the fractal measure interval by imagining a rope with thicker and thinner parts:



Fractal Wave Equation

We define a fractal Laplacian by setting

$$\int u'v'dx = -\int (\Delta_\mu u) v d\mu$$

for all v in the Sobolev space $W^{1,2}$.

This can be thought of as

$$\Delta_\mu u = \frac{d}{d\mu}\frac{d}{dx}u.$$

We study the fractal wave equation

$$\Delta_\mu u = \frac{\partial^2 u}{\partial t^2}$$

with initial conditions $u(x, 0) = \delta(0)$, a Dirac mass at $x = 0$, and $\frac{\partial u}{\partial t}(x, 0) = 0$.

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The First Approach

Using the standard separation of variables approach, if $f_j(x)$ are eigenfunctions of Δ_μ then

$$u(x, t) = \sum_{j=0}^{N} c_j f_j(x) \cos \sqrt{\lambda_j} t$$

solves the wave equation with $\frac{\partial u}{\partial t}(x, 0) = 0$ and $u(x, 0) = \sum_{j=0}^{N} c_j f_j(x)$. For a given N we choose c_j so $u(x, 0)$ best approximates our Dirac mass $\delta(0)$.

Coding the Eigenfunctions

It is possible to compute exact eigenvalues and eigenfunctions of Δ_μ at endpoints of our decomposition of the interval using spectral decimation. By this method one can numerically simulate the Fourier series solution for the wave equation. We used Java code from [1] for this simulation. Here is a snapshot of the wave created by the simulation:

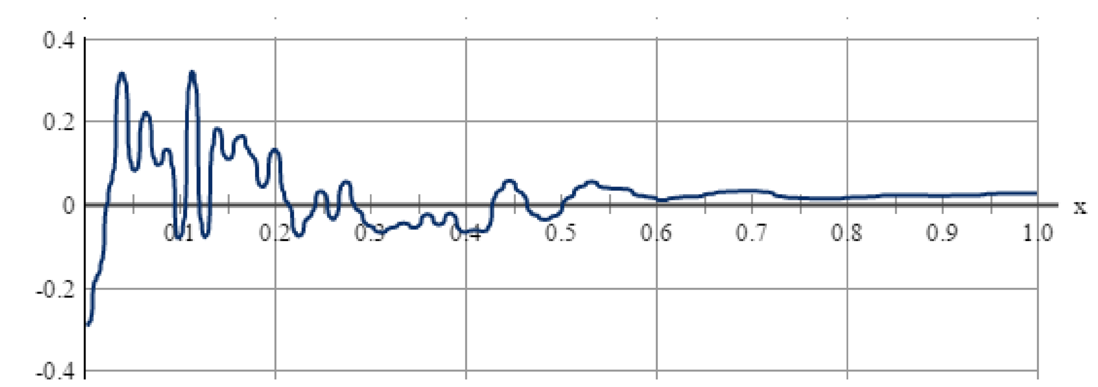


Figure 1: Snapshot $p = 0.2$ and 3^8 subinterval

An advantage of this approach is that the eigenfunctions are those of the limiting fractal measure, not an approximation. The main disadvantage is that we can only approximate the initial Dirac mass pulse.

Wave Propagation Speed

It is known that wave propagation speed over fractal measures is infinite [2]. However, the lower bound on wave amplitude from [2] is exponentially small, so very little of the wave propagates at high speeds We choose a threshold ϵ and study the propagation speed of the ϵ-height wavefront. To do so we define two functions.

Wave envelope:

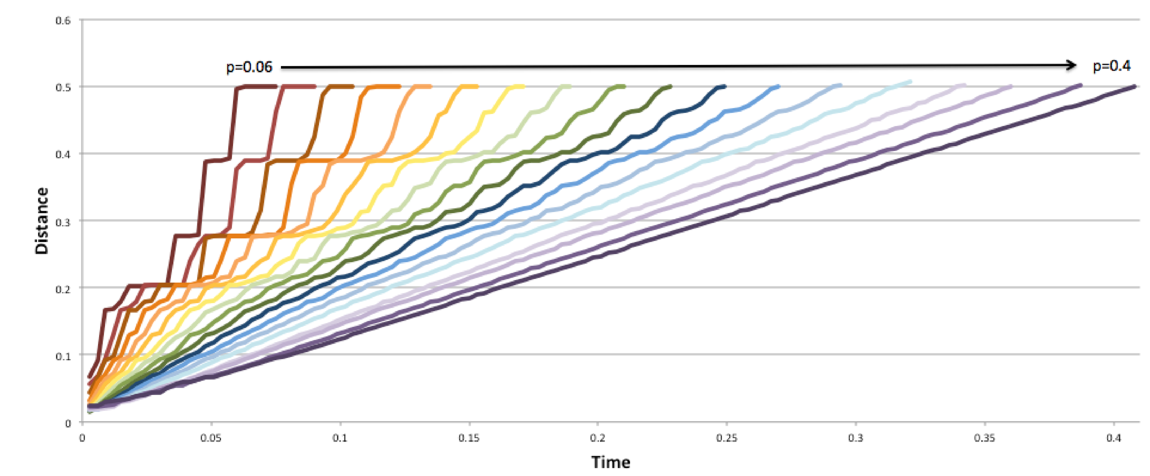
$$U(x, t) = \max\{u(x, s) : s \leq t\}$$

Propagation distance:

$$D(t, \epsilon) = \max\{x : U(x, t) > \epsilon\}$$

Wave Position Data

We studied many features of the functions $U(x, t)$ and $D(t, \epsilon)$, looking for decay properties of the former and at arrival times for the latter. Below is a graph of $D(t, 0.03)$ for different values of p.



Remark 1: We consider only for $x \in [0, 1/2]$ to avoid issues arising from using Neumann eigenfunctions on $[0, 1]$.

Remark 2: We have re-scaled the distance D such that all intervals of our subdivision have the same length 3^{-N} to remove the dependence of the distance structure on p.

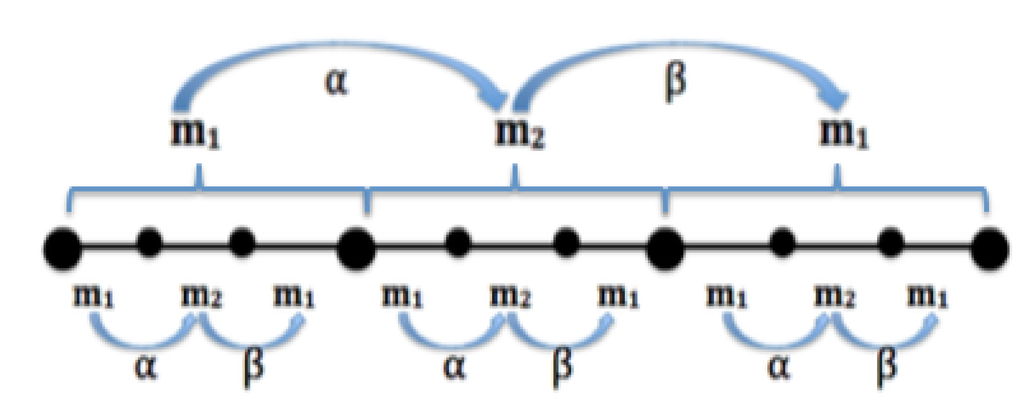
Conjecture 1: $D(t, \epsilon)$ is the integral of a fractal measure.

Conjecture 2: Each $D(t, \epsilon)$ is the same function, but with a time scaling that depends upon p.

The Second Approach

We consider a finite approximation of the fractal by taking a fixed level of the construction of the measure. On each subinterval we then have a classical wave equation with wave speed depending on the interval. At the interfaces between intervals, waves will be transmitted and reflected in proportions depending on the ratios of the wave speeds.

Conveniently, the time for a pulse to traverse an interval is independent of the interval, so we can study the transmissions and reflections with a Markov chain. We call the transition probability for the wave to move from an m_1 to an m_2 subinterval α, and the transition probability to move from an m_2 to an m_1 subinterval β:



Markov simulation

In terms of p and q, $\alpha = 2p$ and $\beta = 2q$. We decompose the wave into left-moving (L) and right-moving (R) parts. To obtain the wave profile at time $T + 1$, we apply the transition matrix S_n to the wave profile at time T. When $n = 1$, the transition matrix S_1 reads

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 1-\beta & 0 \\ 0 & \beta & 0 & 0 & 0 & 1-\alpha \\ 1-\alpha & 0 & 0 & 0 & \beta & 0 \\ 0 & 1-\beta & 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

in the basis $(R_1, R_2, R_3, L_1, L_2, L_3)$.

For general n, the matrix S_n has dimension $2 \cdot 3^n$. Below is a snapshot of the wave profile for $n = 6$ using the Markov chain approach:

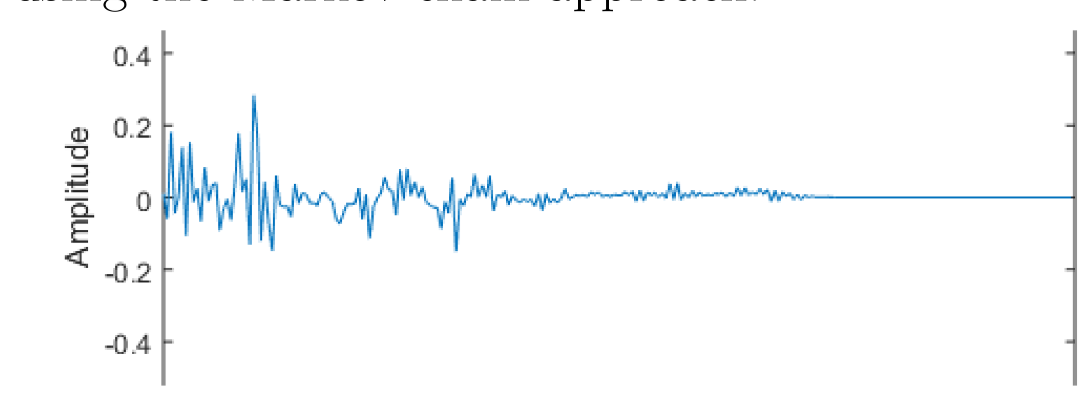
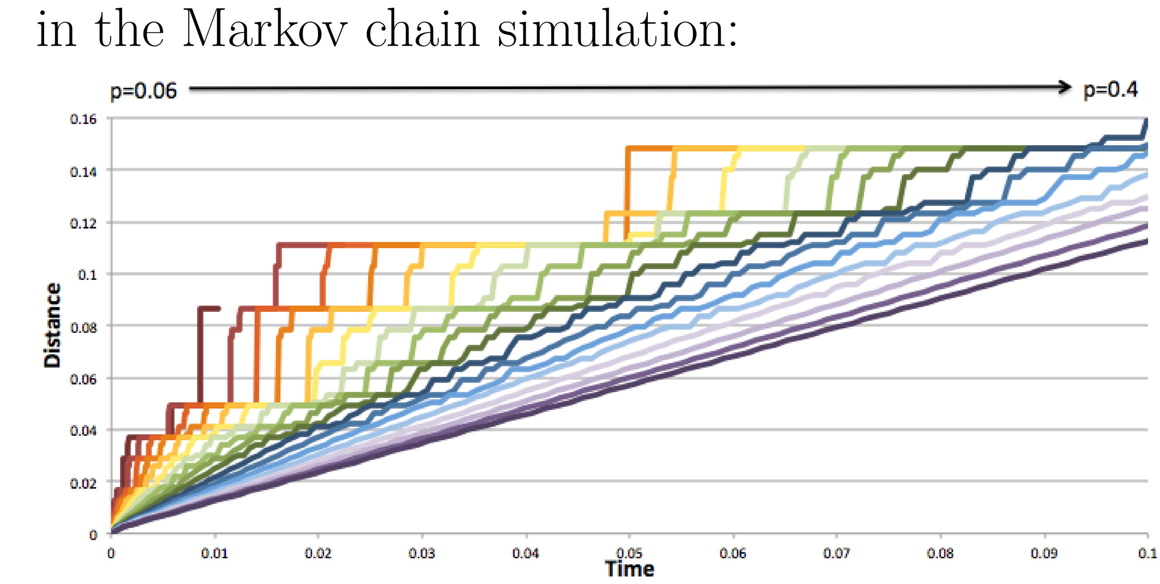


Figure 2: Snapshot with $p = 0.3$ and 3^6 points

Here is a graph of $D(t, \epsilon)$ for different values of p in the Markov chain simulation:



The main advantage of this approach is that the initial condition is a genuine Dirac mass pulse. The main disadvantage is that we work with a finite approximation of the fractal set. Accordingly, the two approaches we have outlined are complementary.

[1] U. Andrews, G. Bonik, J. Chen, R. Martin, and A. Teplyaev *Wave Equation on One-dimensional Fractals with Spectral Decimation.* ArXiv 1505.05855

[2] Y-T. Lee *Infinite Propagation Speed For Wave Solutions on Some P.C.F. Fractals*, ArXiv 1111.2938