Introduction

The study of maximal green sequences (MGS) is motivated by string theory, in particular Donaldson-Thomas invariants and the BPS spectrum. The term maximal green sequences was first introduced by Keller. The definition of an MGS is purely combinatorial and involves transformations of directed graphs known as quivers. We focus on quivers arising from triangulations of polygons. It is known that these quivers are of type A. We find the minimal length of MGS’s for such quivers and develop a procedure that also yields such sequences.

It is necessary to establish some terminology:

A quiver $Q = (Q_0, Q_1)$ is a directed graph where $Q_0$ is a finite set of vertices labeled 1 through $n$, and $Q_1$ is a finite set of arrows. A given vertex $i$ has an arrow $i 	o j$ if there is an arrow from $i$ to $j$. A non-frozen vertex is called green if there are no arrows from a frozen vertex into it. A non-frozen vertex is called red if there are no arrows from it to frozen vertices. Observe that the non-frozen vertices of $Q$ are green.

A green sequence for a quiver $Q$ is a finite sequence of mutations $R_1 \rightarrow R_2 \rightarrow \ldots \rightarrow R_n$, where $R_i$ starts with an arbitrary vertex and then moving in a cyclic order around $R_{i-1}$, the shared vertex $R_i$ is the number of vertices in the quiver and $n$ is the number of vertices in the quiver where every non-frozen vertex is red.

Ex. Consider the quiver $Q$ below mutated at 1.

Given a quiver $Q$, the corresponding framed quiver $\hat{Q} = (Q_0, Q_1)$ by adding a set of frozen vertices $\{v_1, \ldots, n\}$ and a set of arrows $(i \to j \in Q_1)$ to $Q$.

A non-frozen vertex is called green if there are no arrows from frozen vertices into it. A non-frozen vertex is called red if there are no arrows from it to frozen vertices. Observe that the non-frozen vertices of $Q$ are green.

A green sequence for a quiver $Q$ is a finite sequence of mutations $R_1 \rightarrow R_2 \rightarrow \ldots \rightarrow R_n$, where $R_i = Q_0 \cup Q_1$ and $R_0 = \emptyset$. Now consider $Q$ from this point on.

1. Mutate all in $L_0$.
2. Mutate all in $L_0$.
3. Repeat step 3 for every region $L_i$, $i \geq m$.
4. Mutate the vertices of $R_0$ starting with an arbitrary vertex and then moving in a cyclic order around $R_{i-1}$ until the first mutated vertex is mutated again.
5. Mutate at $R_{i-1}$ and then at $L_{i-1}$. Now consider the lower-numbered cycles connected to the vertices of $R_{i-1}$.
6. Repeat the mutations of step 6 for the cycles attached in such a way to $R_{i-1}$.
7. Repeat step 7 for each $T_{ij}$ attached to each $T_{ij} (j > 1)$, which will result in a quiver with vertices that are all red.

As a corollary, we have developed a procedure that produces an MGS of $n + t$ steps for any quiver of type $K$ and a maximal green sequence of length $n$.

Minimal Length: Cyclic

Theorem 1. The following procedure produces an MGS for quivers coming from triangulations of disks consisting entirely of conjugated interior triangles. Moreover, this procedure always consists of $n + t$ mutations, where $n$ is the number of vertices in the quiver and $t$ is the number of 3-cycles.

Definitions.

1. A shared vertex is a vertex that is part of two 3-cycles.
2. A leader is a non-shared vertex that has an arrow to a shared vertex.
3. A follower is a non-shared vertex that has an arrow from a shared vertex.

Let $R_0$ be the full subquiver of $Q$ consisting of a single 3-cycle containing both a leader and a follower. The arrows are of type $K$ but not in $R_0$.

Define the full subquiver $R_1$ of $Q$ to be the union of all 3-cycles containing leaders and followers in $Q$ but not in $R_0$.

Define the full subquiver $R_2$ to be the union of the cycles $T_{ij}$, with leaders and followers in $Q_2 = Q \setminus \{\{i \to j \in Q_1 \mid \{i \to j \} \text{ is leader or follower in } R_1 \} \cup T_0 \cup R_2$.

Define the full subquiver $R_3$ to be the union of the cycles $T_{ij}$, with leaders and followers in $Q_3 = Q \setminus \{\{i \to j \in Q_3 \mid \{i \to j \} \text{ is leader or follower in } R_2 \} \cup T_0 \cup R_3$.

For a cycle $T_{ij}$ labeled the leader $L_{ij}$, the shared vertex $S_{ij}$, and the follower $F_{ij}$.

Theorem 2. The minimal length of a maximal green sequence equivalent to $A$ is $n + t$, where $n$ is the number of vertices in the quiver and $t$ is the number of 3-cycles.

Triangulation Configurations

Maximal Green Sequences and Triangulations of Polygons

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Arbitrary Length: Acyclic Case

We develop procedures to find maximal green sequences of any length given an acyclic quiver with vertices in either a fan or a zigzag configuration. It is known that the minimal length of an MGS is $n$ and that the maximal length of an MGS is $\sum i$, for an acyclic quiver with $n$ vertices.

Facts: We begin by developing a way to find the maximal length MGS’s, which is obtained by mutating at all sinks. We define the minimal length of a MGS to be the union of the vertices in a way that allows for the removal of an arbitrary number of mutations at one particular part of the sequence. Hence, we can find $\delta$ less steps than $\sum i$, where $\delta < k < \sum i$.

Zigzags: Our procedure begins by inducing on the number of vertices of a zigzag with $n = 1$ vertices and adding one extra mutation step for the sub quiver to the known MGS’s for the quiver with $n = 1$ vertices. However, the inductive step does not give all possible length MGS’s for a zigzag with $n = 1$ vertices. Thus, to find the remaining MGS’s, we develop a maximal length procedure and use a subtractive technique that is similar to the ones used in the fan case.

Maximal Length Conjectures

Given a quiver $Q$ composed of conjugated 3-cycles and $n$ vertices, the length of the longest MGS is bounded above by $\sum i$, but is not generally equal to $\sum i$. Furthermore, the longest length of an MGS for $Q$ varies depending on the configuration of shared vertices. These two observations make the problem of finding a longest MGS for a given $Q$ more difficult to solve. We chose to approach the problem of determining these lengths computationally.

We wrote a program in Matlab which computes all maximal green sequences for a given quiver and records each sequence up to commutative mutations. Using this data, we were able to develop a procedure for finding these longest length sequences. We employed this procedure to gather data sets on longest length MGS’s for conjugated 3-cycles. We proceeded to analyze this data and created equations which model the length of a longest MGS for a quiver of conjugated 3-cycles based on the underlying structure of shared vertices.

Maximal MGS length (M) formulas for varying shared vertex configurations composed of t triangles:

For two fans meeting at the shared vertex,

$M = t^2 + 3 + 2t + t^2 + n - 1$

For zigzags with odd $t$:

$M = t^2 + 2 + \frac{1}{2}t - 1$

For source zigzags with even $t$:

$M = t^2 + 2 + \frac{1}{2}t - 1$

For sink zigzags with even $t$:

$M = t^2 + 2 + \frac{1}{2}t - 1$

References


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