Maximal Green Sequences and Triangulations of Polygons

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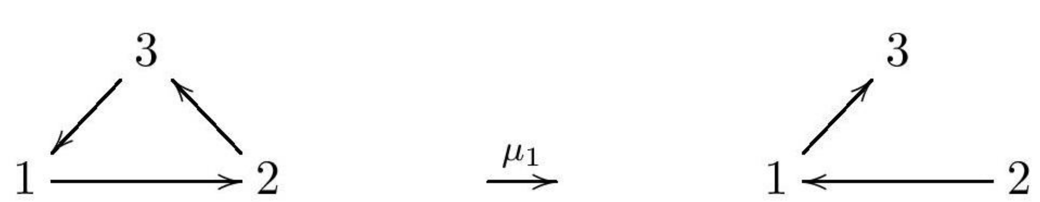
Introduction

The study of maximal green sequences (MGS) is motivated by string theory, in particular Donaldson-Thomas invariants and the BPS spectrum. The term maximal green sequences was first introduced by Keller. The definition of an MGS is purely combinatorial and involves transformations of directed graphs known as quivers. We focus on quivers arising from triangulations of polygons. It is known that these quivers are of type $\mathbb{A}$. We find the minimal length of MGS's for such quivers and develop a procedure that also yields such sequences.

It is necessary to establish some terminology:

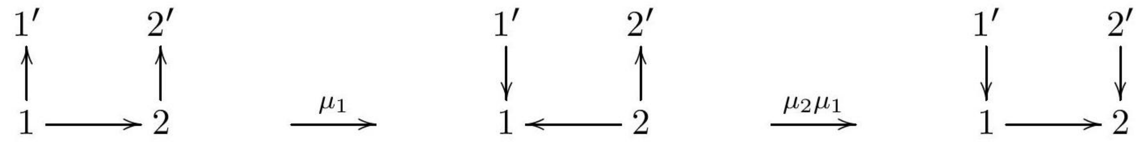
- A *quiver* $Q = (Q_0, Q_1)$ is a directed graph where Q_0 is a finite set of vertices labeled 1 through n, and Q_0 is a finite set of arrows.
- Given a vertex $i \epsilon Q_0$, the *mutation* of a quiver Q at i, denoted $\mu_i(Q)$, is a new quiver formed by applying the following steps to Q:
  1. For any pair of arrows $h \to i \to j$ in Q , add a new arrow $h \to j$.
  2. Reverse all arrows incident to i.
  3. Remove a maximal collection of oriented 2-cycles.

Ex. Consider the quiver Q below mutated at 1.



- Given a quiver Q, construct the corresponding *framed quiver* $\hat{Q} = (\widehat{Q_0}, \widehat{Q_1})$ by adding a set of frozen vertices $\{1', 2', \dots, n'\}$ and a set of arrows $\{i \to i' \mid i \in Q_0\}$ to Q.
- A non-frozen vertex i is called *green* if there are no arrows from frozen vertices into i. A non-frozen vertex j is called *red* if there are no arrows from j into frozen vertices. Observe that the non-frozen vertices of $\hat{Q}$ are green.
- A *green sequence* for a quiver Q is a finite sequence of mutations $\mu_{ij} \dots \mu_{i2}\mu_{i1}$ of $\hat{Q}$ such that each consecutive mutation is performed in a green vertex of the corresponding quiver. A *maximal green sequence* (MGS) for Q is a green sequence that transforms $\hat{Q}$ into a quiver where every non-frozen vertex is red.

Ex. The quiver $Q = 1 \to 2$ has an MGS $\mu_2\mu_1$.



- Given a polygon P with a set of nodes K, an *arc* γ in P is a smooth non self-crossing curve that has endpoints in K, and is otherwise disjoint from K. Moreover, we only consider curves up to homotopy.
- A *triangulation* T of a polygon P is a maximal collection of non-intersecting arcs (refer to Figure 1).

There is a construction that yields a quiver from a given triangulation. Arcs in a triangulation T correspond to vertices in the quiver, and angles between arcs correspond to paths in the quiver. A quiver Q_T arising in this way from a triangulation T of a polygon, can either be acyclic, or contain 3-cycles.

Minimal Length: Acyclic

It is known that acyclic quivers admit MGS's of minimal length n, where n is the number of vertices in the quiver. Such an MGS can be obtained by mutating only at sources (vertices with no arrows coming into them) until each vertex has been mutated exactly once.

Triangulation Configurations

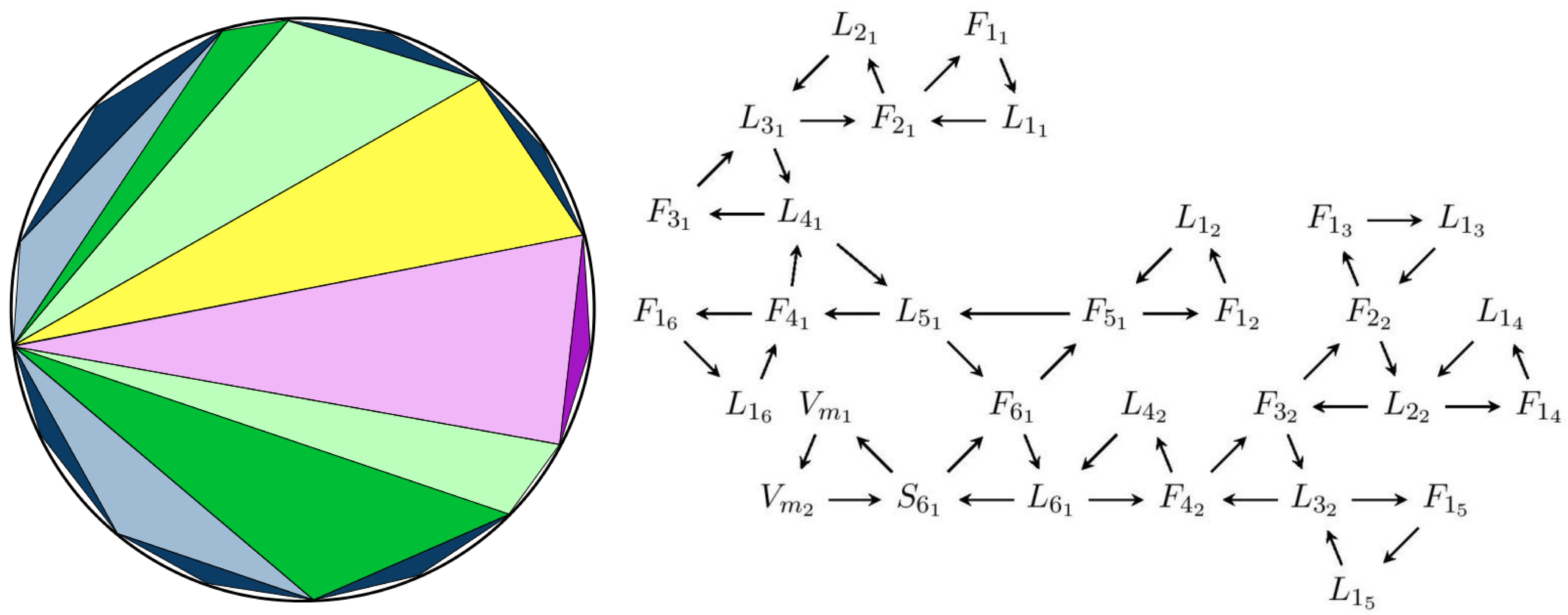


Figure 1. (Left) This figure shows a triangulation composed of interior triangles. Dark Blue=R_1, Light Blue=R_2, Dark Green=R_3, Light Green=R_4, Yellow=R_5, Light Purple=R_6, Dark Purple=R_m. (Right) Corresponding quiver diagram with labeling for the triangulation on the left.

Minimal Length: Cyclic

Theorem 1. *The following procedure produces an MGS for quivers coming from triangulations of disks consisting entirely of conjoined interior triangles. Moreover, this procedure always consists of $n + t$ mutations, where n is the number of vertices in the quiver and t is the number of 3-cycles.*

1. Consider Q_T (or simply Q). Establish $R_1, R_2, \dots, R_m$ as outlined in Definitions $1 - 7$ below. Label the vertices of R_m as $V_{m_1}, V_{m_2}, S_{m'_1}$, where $S_{m'_1} \in R_{m'}$. Now consider $\widehat{Q}$ from this point on.
2. Mutate in all L_{1_i}.
3. Mutate in all L_{2_i}.
4. Repeat step 3 for every region R_i, $i \leq m'$.
5. Mutate the vertices of R_m starting with an arbitrary vertex and then moving in a cyclic order around R_m, until the vertex that was first mutated is mutated again.
6. Mutate at $F_{m'_1}$ and then at $L_{m'_1}$. Now consider the lower-numbered cycles connected to the vertices of $T_{m'_1}$.
7. Repeat the mutations of step 6 for the cycles attached in such a way to $R_{m'}$.
8. Repeat step 7 for each T_{i_k} attached to each T_{j_l} $(j > i)$, which will result in a quiver with vertices that are all red.

As a corollary, we have developed a procedure that produces an MGS of $n + t$ steps for any quiver of type $\mathbb{A}$.

Definitions.
1. A *shared vertex* is a vertex that is part of two 3-cycles.
2. A *leader* is a non-shared vertex that has an arrow to a shared vertex.
3. A *follower* is a non-shared vertex that has an arrow to a leader.
4. Let R_m be the full subquiver of Q consisting of a single 3-cycle containing both a leader and a follower.
5. Define the full subquiver R_1 of Q to be the union of all 3-cycles containing leaders and followers in Q but not in R_m.
6. Define the full subquiver R_2 to be the union of the cycles $T_{2_1}, \dots, T_{2_m}$ with leaders and followers in $Q_2 \coloneqq Q \setminus \{V \in R_1 \mid V \text{ is a leader or follower in } R_1\}$ with $T_m \notin R_2$.
7. Define the full subquiver R_i to be the union of the cycles $T_{i_1}, \dots, T_{i_r}$ with leaders and followers in $Q_i \coloneqq Q_{i-1} \setminus \{V \in R_{i-1} \mid V \text{ is a leader or follower in } R_{i-1}\}$ with $T_m \notin R_i$. For a cycle T_{i_j}, label the leader L_{i_j}, the shared vertex S_{i_j}, and the follower F_{i_j}.

Theorem 2. *The minimal length of a maximal green sequence for a quiver mutation equivalent to $\mathbb{A}$ is $n + t$, where n is the number of vertices in the quiver and t is the number of 3-cycles.*

Arbitrary Length: Acyclic Case

We develop procedures to find maximal green sequences of any length given an acyclic quiver with n vertices in either a fan or a zigzag configuration. It is known that the minimal length of an MGS is n and that the maximal length of an MGS is $\sum_{i=1}^{n} i$, for an acyclic quiver with n vertices.

Fans: We begin by developing a way to find the maximal length MGS's, which is attained by mutating at all sinks. We define the mutation sequence of sink vertices in a way that allows for the removal of an arbitrary number of mutation steps at one particular part of the sequence. Hence, we can find k less steps than $\sum_{i=1}^{n} i$, where $0 < k < \sum_{i=1}^{n-1} i$.

Zigzags: Our procedure begins by inducting on the number of vertices of a zigzag with $n - 1$ vertices and adding one extra mutation step for the nth vertex to the known MGS's for the quiver with $n - 1$ vertices. However, the inductive step does not give all possible length MGS's for a zigzag with n vertices. Thus, to find the remaining MGS's, we develop a maximal length procedure and use a subtractive technique that is similar to the one used in the fan case.

Maximal Length Conjectures

Given a quiver Q composed of conjoined 3-cycles and n vertices, the length of the longest MGS is bounded above by $\sum_{i=1}^{n} i$, but is not generally equal to $\sum_{i=1}^{n} i$. Furthermore, the longest length of an MGS for Q varies depending on the configuration of shared vertices. These two observations make the problem of finding a longest MGS for a given Q more difficult to solve. We chose to approach the problem of determining these lengths computationally.

We wrote a program in Matlab which computes all maximal green sequences for a given quiver and records each sequence up to commutative mutations. Using this data, we were able to develop a procedure for finding these longest length sequences. We employed this procedure to gather data sets on longest length MGSs for conjoined 3-cycles. We proceeded to analyze this data and created equations which model the length of a longest MGS for a quiver of conjoined 3-cycles based on the underlying structure of shared vertices.

Maximal MGS length (M) formulas for varying shared vertex configurations composed of t triangles:

For two fans meeting at the s shared vertex,

$$M = t^2 + 3t + 2st - s^2 + 1$$

For zigzags with odd t: $M = \frac{9}{8}t^2 + \frac{9}{2}t - \frac{5}{8}$

For source zigzags with even t: $M = \frac{9}{8}t^2 + \frac{21}{4}t - 2$

For sink zigzags with even t: $M = \frac{9}{8}t^2 + \frac{15}{4}t + 1$

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