Stabilization by Noise in $\mathbb{C}^2$–Valued Nonlinear Systems

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Abstract

We consider a system of $\mathbb{C}^2$–valued ordinary differential equations (ODE) that has solutions which blow up in finite time. By adding a Brownian stochastic term, we can stabilize the entire system and produce a statistical equilibrium for all solutions.

Introduction

The following $\mathbb{C}^2$–valued system has solutions which blow up in finite time:

$$\begin{cases} \dot{z} = -\nu z + \alpha z w \\ \dot{w} = -\nu w + \beta z w \\ z(0) = z_0 \in \mathbb{C} \\ w(0) = w_0 \in \mathbb{C}, \end{cases}$$

where $\nu \in \mathbb{R}^+$ and $\alpha, \beta \in \mathbb{R}$. Recent results have shown how an additive Brownian motion can stabilize complex single–variable polynomial systems [2]. We have numerical results that suggest the same results can be produced for our system. We analytically identified which initial conditions produce unstable solutions. We then verified numerically that the addition of Brownian motion stabilized these formerly unstable solutions.

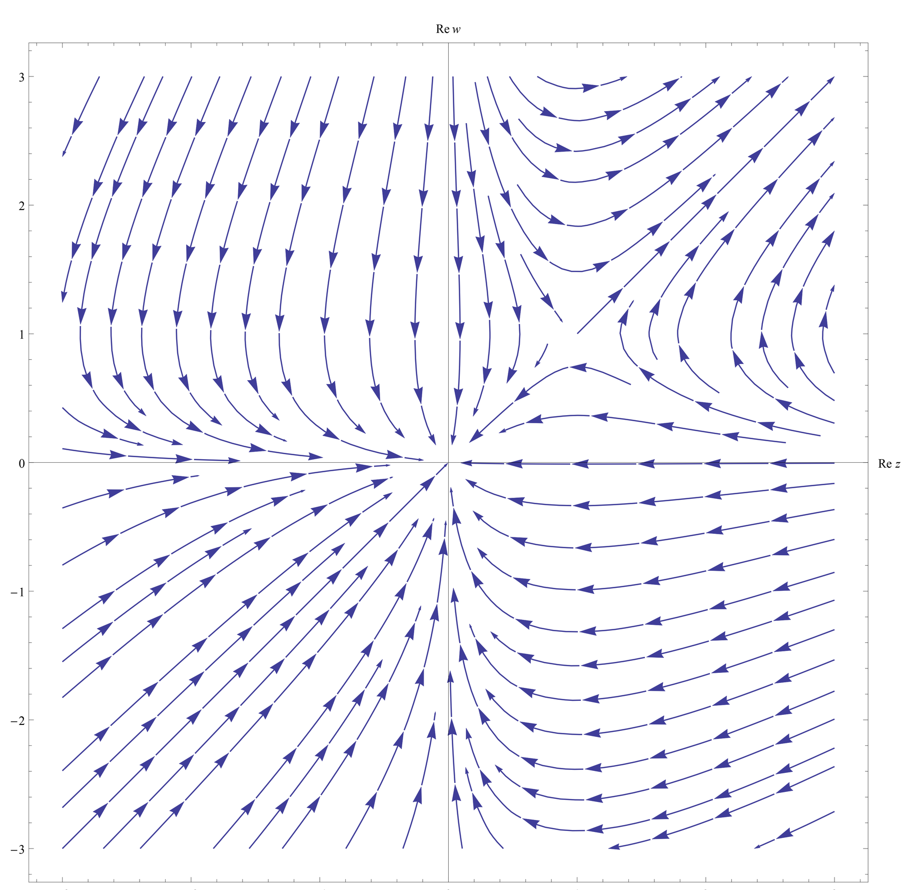

Figure 1: Dynamics in the Real Plane

Background

A more thorough understanding of our system may be achieved by considering the following polynomial system, [2]:

$$\dot{z}_t = a_{n+1} z_t^{n+1} + a_n z_t^n + \cdots + a_0, \quad n \geq 1$$

Like our multivariable system, this ODE has unstable trajectories that reach infinity in finite time. These unstable trajectories can be stabilized by the addition of a Brownian motion term which gives us the following stochastic differential equation (SDE):

$$\begin{cases} dz_t = (a_{n+1} z_t^{n+1} + a_n z_t^n + \cdots + a_0)\, dt + \sigma dB_t \\ z_0 \in \mathbb{C}, \end{cases}$$

where B_t is a complex Brownian stochastic term and $\sigma \in \mathbb{R}^+$. We would like to extend the methods developed in [2] to show that our system can be stabilized by the addition of an appropriate Brownian motion and has an invariant measure.

Objectives

- Identify regions with unstable solutions in our $\mathbb{C}^2$–valued system of equations
- Show what kind of noise is needed to stabilize our system

Analytical Observations

After decoupling $\dot{z}$ and $\dot{w}$ and making a change of coordinates, we get the following system of equations:

$$\begin{cases} \dot{y}_1 = -\nu y_1 + \beta[(y_1^2 - y_2^2) - (y_3^2 - y_4^2)] \\ \dot{y}_2 = -\nu y_2 \\ \dot{y}_3 = -\nu y_3 + 2\beta(y_1 y_3 - y_2 y_4) \\ \dot{y}_4 = -\nu y_4 \end{cases}$$

This system can be further simplified to:

$$\begin{cases} \dot{y}_1 = -\nu y_1(t) + \beta(y_1^2(t) - y_3^2(t)) - \beta(y_2^2(0) - y_4^2(0)) e^{-2\nu t} \\ \dot{y}_3 = -\nu y_3(t) + 2\beta y_1(t) y_3(t) - 2\beta y_2(0) y_4(0) e^{-2\nu t}. \end{cases}$$

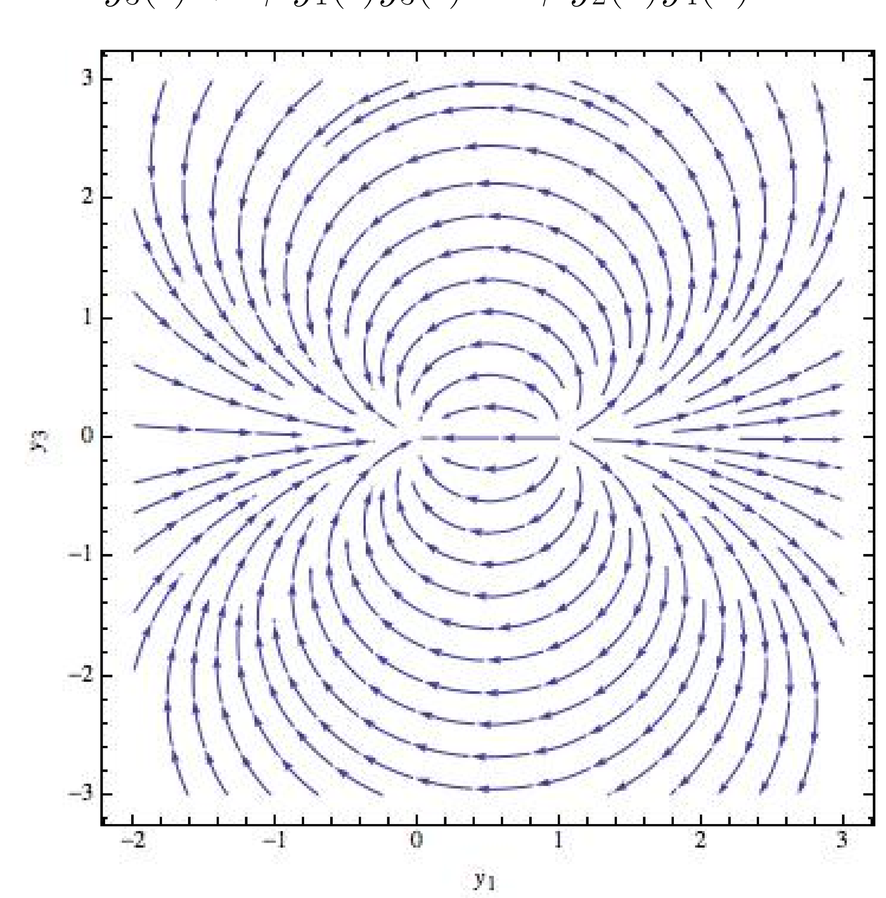

Figure 2: Phase portrait for large t (looks like $\dot{z} = z^2 - z$)

Numerical Results

For our system, we can stabilize the explosive solutions by adding Brownian motion in the imaginary $\dot{z}$ direction:

$$\begin{cases} dz_t = (-\nu z + \alpha z w)\, dt + \mathbf{idB_t} \\ dw_t = (-\nu w + \beta z w)\, dt \\ z(0) = z_0 \in \mathbb{C} \\ w(0) = w_0 \in \mathbb{C}, \end{cases}$$

In the transformed coordinates, we have the following:

$$\begin{cases} dy_1 = (-\nu y_1 + \beta[(y_1^2 - y_2^2) - (y_3^2 - y_4^2)])\, dt \\ dy_2 = (-\nu y_2)\, dt \\ dy_3 = (-\nu y_3 + 2\beta(y_1 y_3 - y_2 y_4))\, dt + \frac{1}{2}\mathbf{dB_t} \\ dy_4 = (-\nu y_4)\, dt + \frac{1}{2}\mathbf{dB_t}. \end{cases}$$

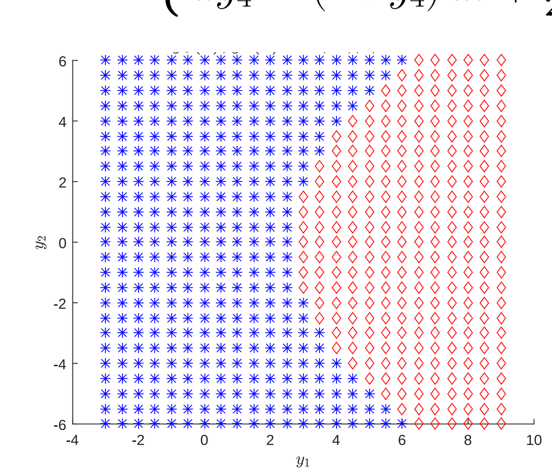

Figure 3: $y_3(0), y_4(0) = 0$, without noise

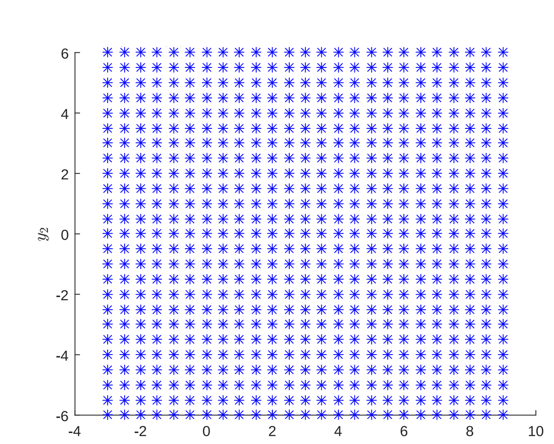

Figure 4: $y_3(0), y_4(0) = 0$, with noise

$\diamond$ = explosive solutions

$*$ = non–explosive solutions

Conclusion

We ran numerical simulations to verify our analytical results that this SDE must be stable. The random noise was simulated by adding a normally distributed random variable to each time-step of the iterated Euler's method. This computation was done on the same set of initial conditions to ensure that the probability of a stable solution curve is near 1. We tested a large range of initial conditions to ensure that this SDE is stable everywhere in the 4-dimensional space, for all valid α, β, and ν.

Continued Research

- How can we generalize our results to systems in higher dimensions and with higher order nonlinearity (ie. cubic, quartic, etc.)?
- If ν is negative, or if two distinct constants ν_1 , ν_2 are used, how does the system differ and what classes of noise will stabilize the system?

References

[1] David P. Herzog. Geometry's fundamental role in the stability of stochastic differential equations. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)-The University of Arizona.

[2] David P. Herzog and Jonathan C. Mattingly. Noise-induced stabilization of planar flows I. *arXiv:1404.0957*, 2014.

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