Maximal Green Sequences for Triangulations of Polygons

Minimal Length Maximal Green Sequences for Type $\mathbb{A}$ Quivers

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Motivation

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- The definition of an MGS is purely combinatorial and involves transformations of directed graphs known as quivers.

Quivers

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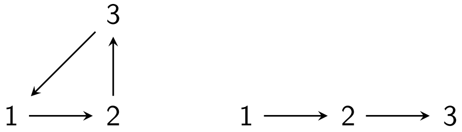
Ex.



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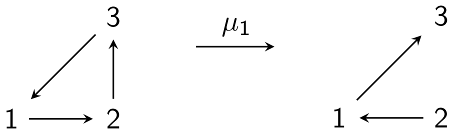

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Ex. Suppose $Q : 1 \longrightarrow 2$

$$\hat{Q} : \begin{array}{ccc} 1' & & 2' \\ \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 \end{array}$$

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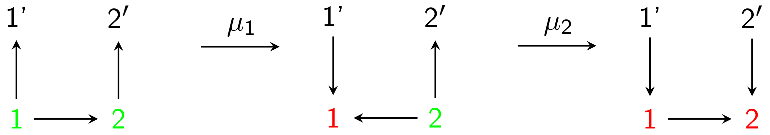

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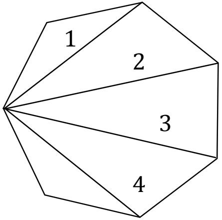
T :



$Q_T : 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$

Acyclic Quivers

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Proposition

A minimal length MGS for an acyclic quiver Q can be obtained by mutating at sources until each vertex has been mutated exactly once. This procedure yields an MGS of minimal length n, where n is the number of vertices in Q.

Definitions

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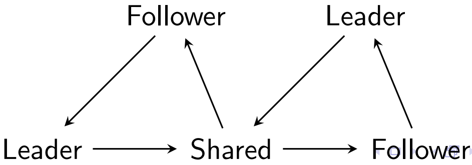
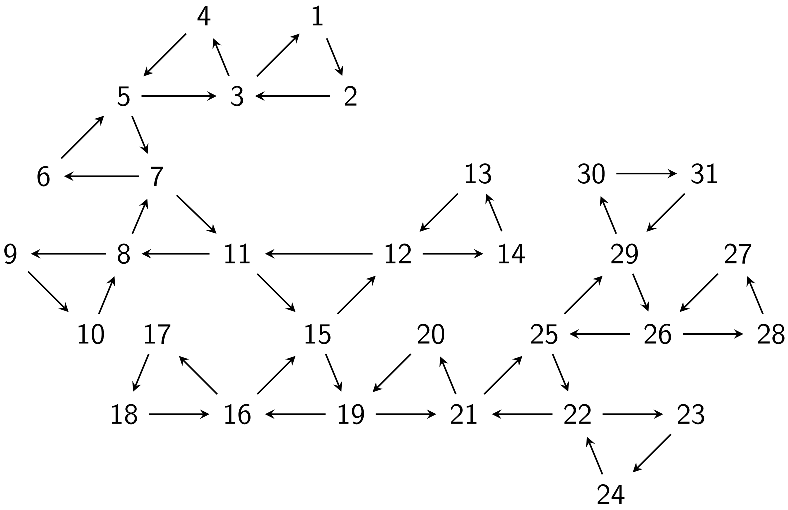


Labeling


4
1
5
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6
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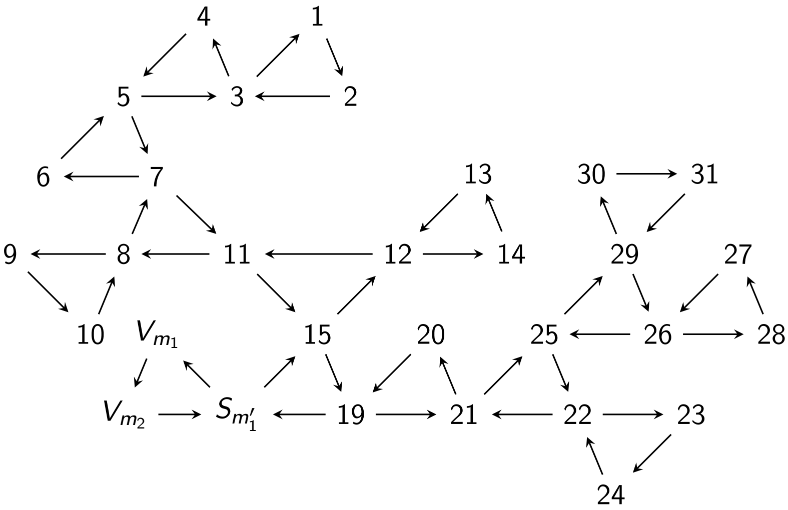
Innermost Region



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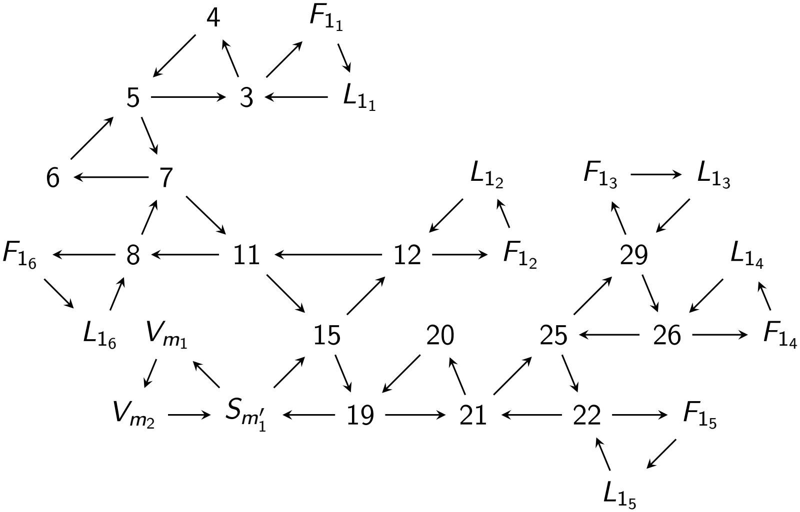
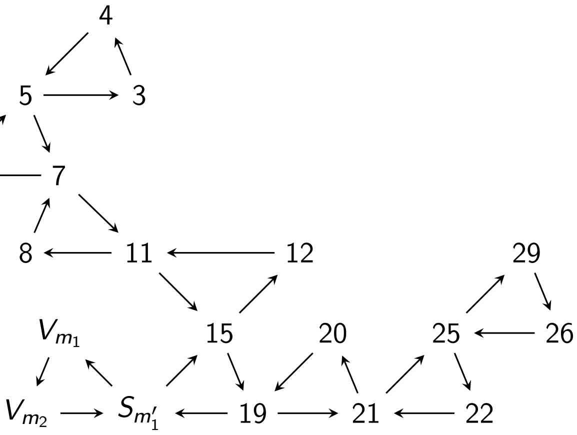


Region 2



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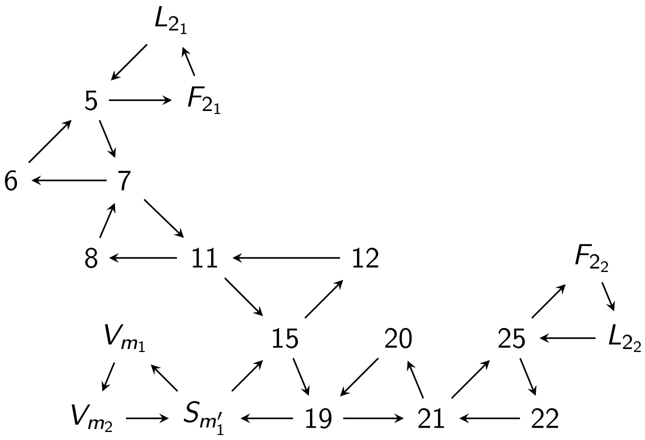

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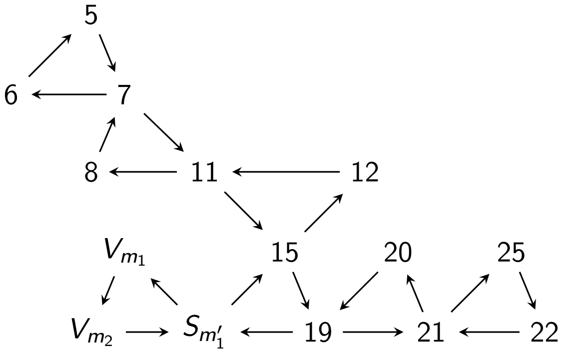

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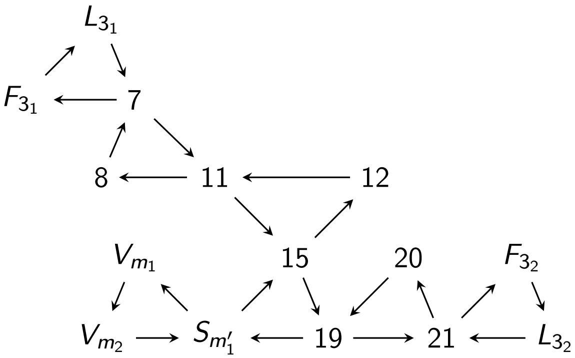
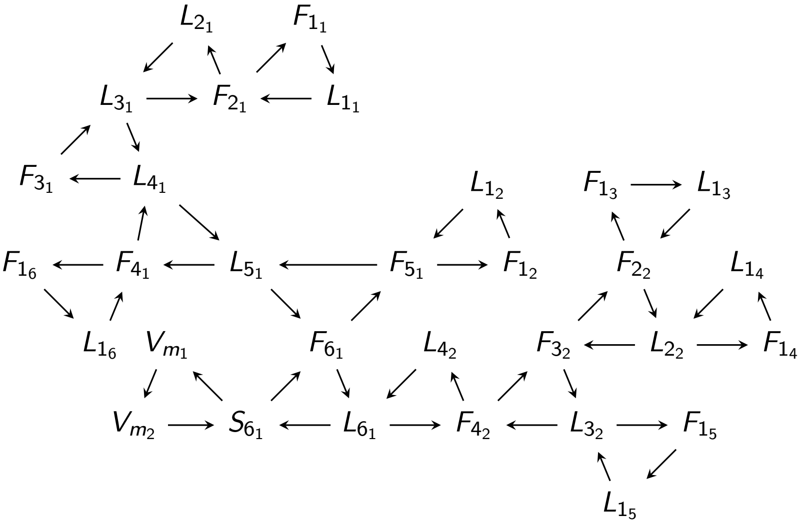


Example Quiver



Main Theorem

Theorem 1

The following procedure produces an MGS for quivers coming from triangulations of disks consisting entirely of conjoined interior triangles. Moreover, this procedure always consists of $n + t$ mutations, where n is the number of vertices in the quiver and t is the number of 3-cycles.

Procedure

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1. Consider Q_T (or simply Q). Establish $R_1, R_2, ..., R_m$ as outlined in Definitions $4 - 7$. Label the vertices of R_m as $V_{m_1}, V_{m_2}, S_{m'_1}$, where $S_{m'_1} \in R_{m'}$. Now consider $\hat{Q}$ from this point on.

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6. Mutate at $F_{m'_1}$ and then at $L_{m'_1}$. Call this mutation sequence $\underline{\mu_{m'_1}}$. Now consider the lower-numbered cycles connected to the vertices of $T_{m'_1}$.

Procedure

7. Repeat the mutations of step 6 for the cycles attached in such a way to $R_{m'}$.

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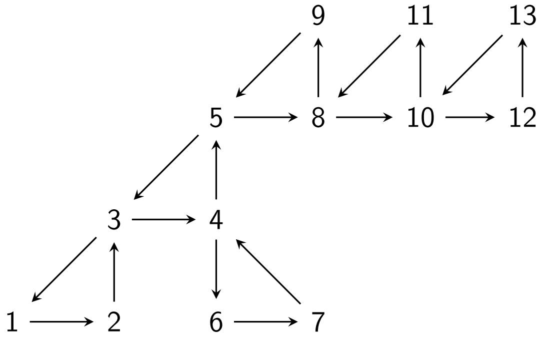
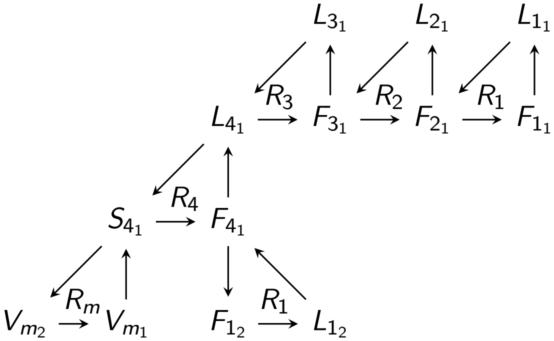
8. Repeat step 7 for each T_{i_k} attached to each T_{j_k} $(j > i)$, which will result in a quiver with vertices that are all red.

Example



Example


L_{3_1} L_{2_1} L_{1_1}
R_3 R_2 R_1
L_{4_1} F_{3_1} F_{2_1} F_{1_1}
R_4
S_{4_1} F_{4_1}
R_m R_1
V_{m_2} V_{m_1} F_{1_2} L_{1_2}

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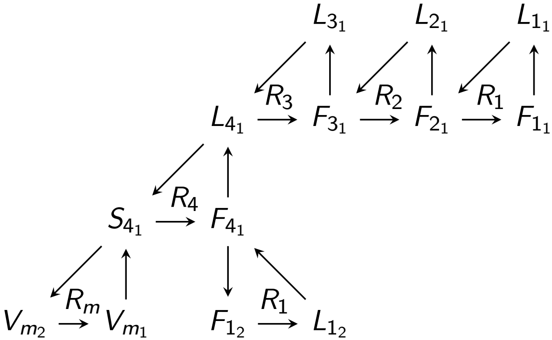
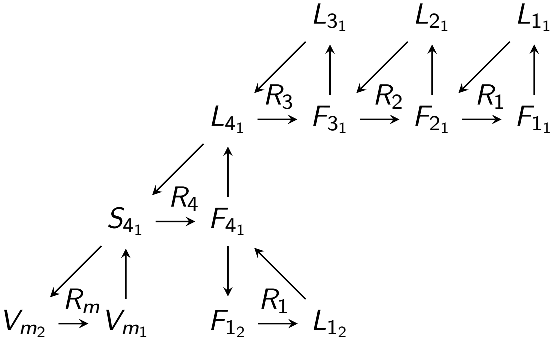


Mutation Sequence:

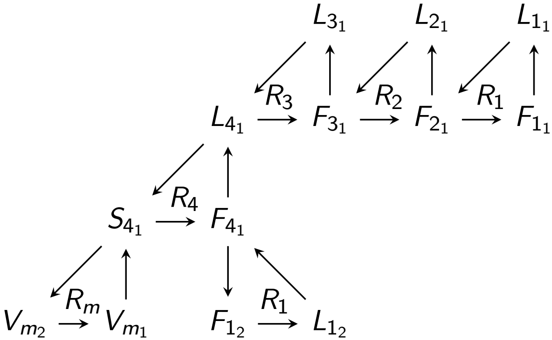
Example



Mutation Sequence:

$\mu_{L_{1_2}} \mu_{L_{1_1}}$

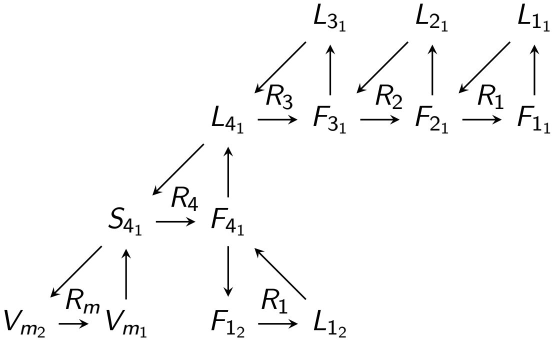
Example



Mutation Sequence:

$\mu_{L_{2_1}} \mu_{L_{1_2}} \mu_{L_{1_1}}$

Example



Mutation Sequence:

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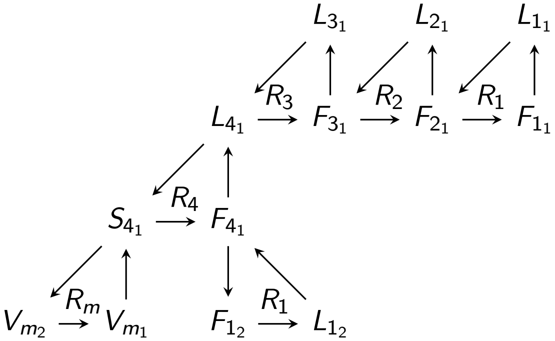
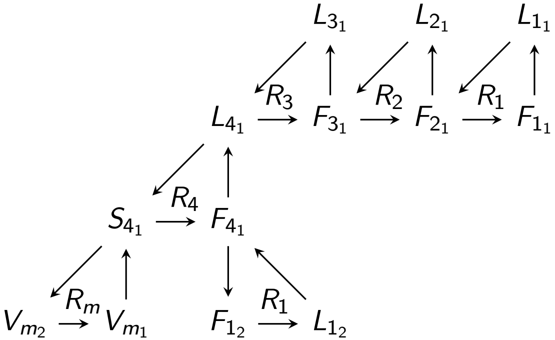
Example



Mutation Sequence:

$\mu_{L_{4_1}} \mu_{L_{3_1}} \mu_{L_{2_1}} \mu_{L_{1_2}} \mu_{L_{1_1}}$

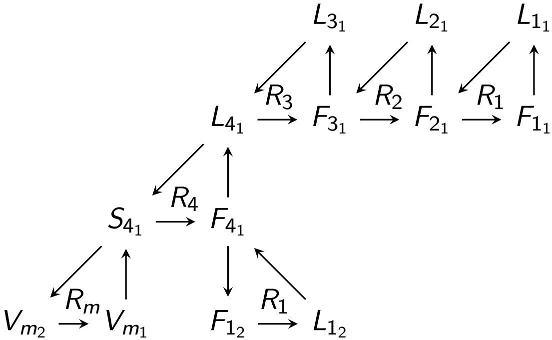
Example



Mutation Sequence:

$$\mu_{S_{4_1}}\mu_{V_{m_2}}\mu_{V_{m_1}}\mu_{S_{4_1}}\mu_{L_{4_1}}\mu_{L_{3_1}}\mu_{L_{2_1}}\mu_{L_{1_2}}\mu_{L_{1_1}}$$

Example



Mutation Sequence:

$$\mu_{L_{4_1}} \mu_{F_{4_1}} \mu_{S_{4_1}} \mu_{V_{m_2}} \mu_{V_{m_1}} \mu_{S_{4_1}} \mu_{L_{4_1}} \mu_{L_{3_1}} \mu_{L_{2_1}} \mu_{L_{1_2}} \mu_{L_{1_1}}$$

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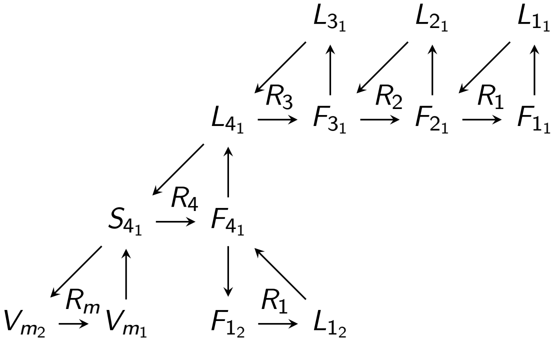

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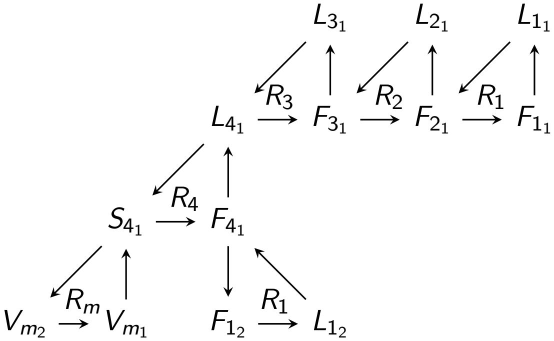

Mutation Sequence:

$$\mu_{L_{1_2}} \mu_{F_{1_2}} \mu_{L_{3_1}} \mu_{F_{3_1}} \mu_{L_{4_1}} \mu_{F_{4_1}} \mu_{S_{4_1}} \mu_{V_{m_2}} \mu_{V_{m_1}} \mu_{S_{4_1}} \mu_{L_{4_1}} \mu_{L_{3_1}} \mu_{L_{2_1}} \mu_{L_{1_2}} \mu_{L_{1_1}}$$

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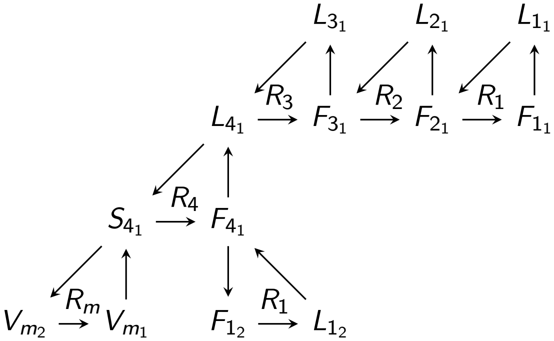

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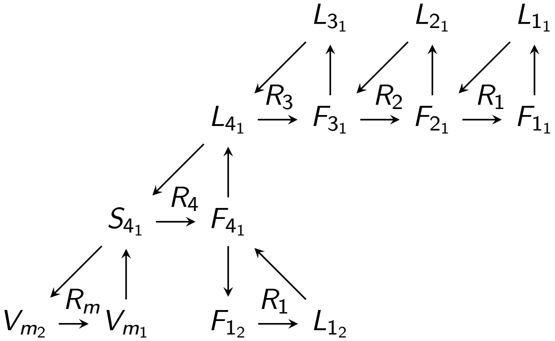
Example



Mutation Sequence:

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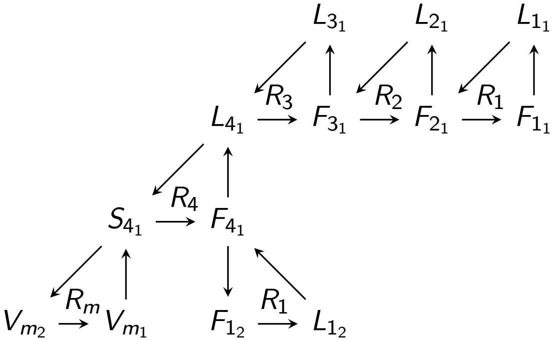
Example



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Notice that this MGS has length $19 = 13 + 6 = n + t$, as desired.

Preparatory Definitions for the General Procedure

A **non-isolating vertex** is a vertex in an acyclic subquiver with an arrow from itself into a vertex in a 3-cycle. An **isolating vertex** is a vertex in an acyclic subquiver with an arrow going into it from a vertex in a 3-cycle.

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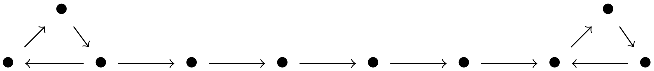
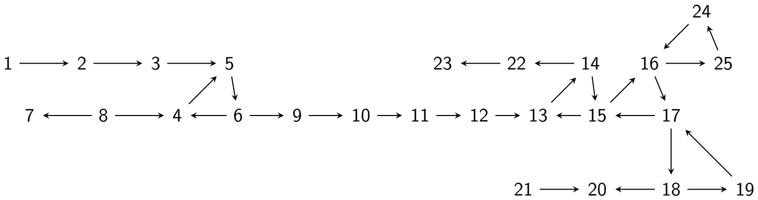
Call a fan containing both an isolating and non-isolating vertex, as shown below, a **connecting fan**.

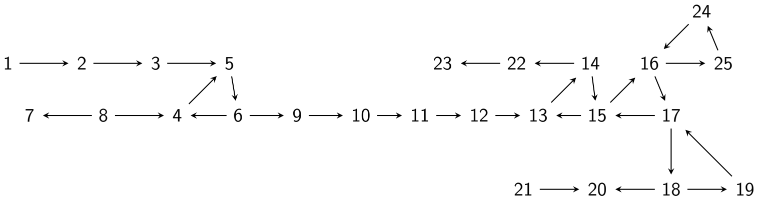


The General Procedure



1) Choose a 3-cycle configuration in Q which is not led into by a connecting fan, and label it C_0.

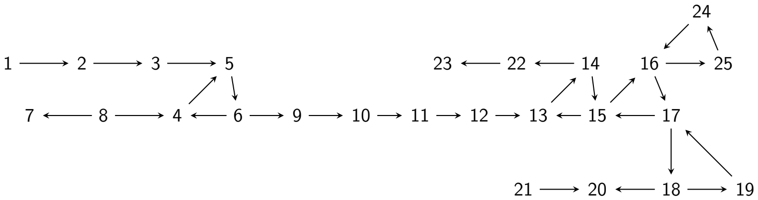
The General Procedure



1) Choose a 3-cycle configuration in Q which is not led into by a connecting fan, and label it C_0.

2) Isolate C_0 via mutations at source vertices in adjacent acyclic subquivers.

The General Procedure

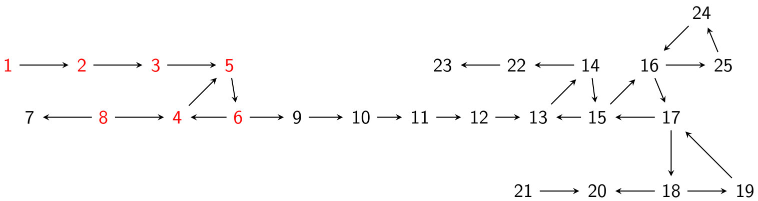


1) Choose a 3-cycle configuration in Q which is not led into by a connecting fan, and label it C_0.

2) Isolate C_0 via mutations at source vertices in adjacent acyclic subquivers.

3) Resolve C_0 via the 3-cycle procedure.

The General Procedure



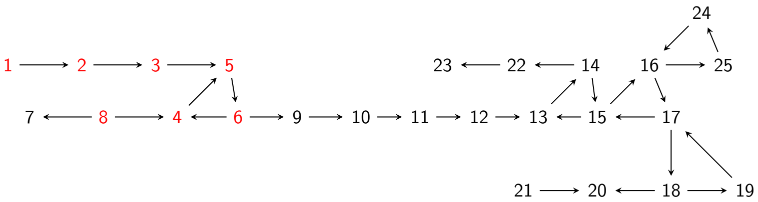
1) Choose a 3-cycle configuration in Q which is not led into by a connecting fan, and label it C_0.

2) Isolate C_0 via mutations at source vertices in adjacent acyclic subquivers.

3) Resolve C_0 via the 3-cycle procedure.

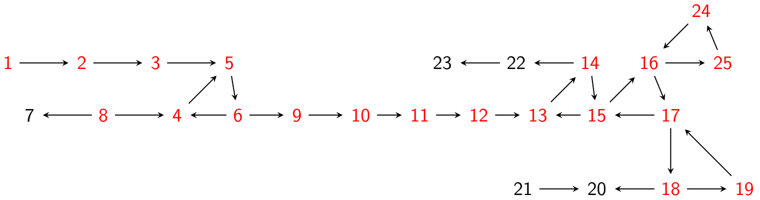
4) Consider the subquiver Q_1 composed of all remaining green vertices in Q.

The General Procedure



1) Choose a 3-cycle configuration in Q which is not led into by a connecting fan, and label it C_0.
2) Isolate C_0 via mutations at source vertices in adjacent acyclic subquivers.
3) Resolve C_0 via the 3-cycle procedure.
4) Consider the subquiver Q_1 composed of all remaining green vertices in Q.
5) Choose C_1 analogously to C_0, and repeat steps 2-4.

The General Procedure



1) Choose a 3-cycle configuration in Q which is not led into by a connecting fan, and label it C_0.

2) Isolate C_0 via mutations at source vertices in adjacent acyclic subquivers.

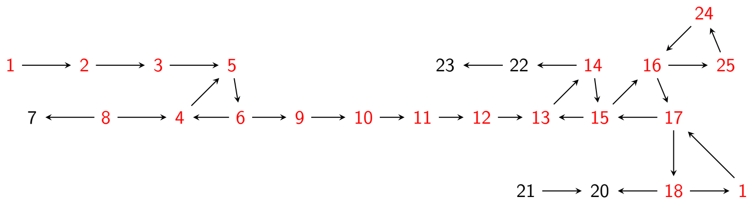
3) Resolve C_0 via the 3-cycle procedure.

4) Consider the subquiver Q_1 composed of all remaining green vertices in Q.

5) Choose C_1 analogously to C_0, and repeat steps 2-4.

6) Continue until all 3-cycles are resolved.

The General Procedure



1) Choose a 3-cycle configuration in Q which is not led into by a connecting fan, and label it C_0.
2) Isolate C_0 via mutations at source vertices in adjacent acyclic subquivers.
3) Resolve C_0 via the 3-cycle procedure.
4) Consider the subquiver Q_1 composed of all remaining green vertices in Q.
5) Choose C_1 analogously to C_0, and repeat steps 2-4.
6) Continue until all 3-cycles are resolved.
7) Mutate any remaining green vertices via the procedure for acyclic quivers.

Thank you.