Magnetic Spectral Decimation on the Diamond Fractal

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Outline

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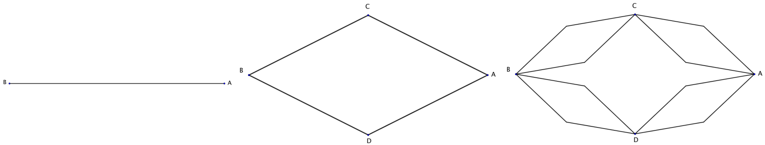
Motivation

Understanding the spectrum allows us to analyze the energy levels of a charged particle confined to the diamond fractal and under the influence of a magnetic field.

The Diamond Fractal



- Each edge in the n^{th} level approximating graph becomes a diamond in the $(n+1)^{\text{th}}$ level.
- We can represent this operation using an iterated function system of four contraction maps f_i.
- The diamond fractal is the unique, non-empty, compact set invariant under the IFS: $K = \bigcup f_i(K)$.

Laplacian Matrix

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- The graph Laplacian at level n is an operator on functions given by

$$\Delta_n u(x) = \sum_{y \backsim x} (u(x) - u(y))$$

- This can be represented as the difference of the graph's degree matrix and adjacency matrix. At level zero we have:

Degree Matrix	Adjacency Matrix	Laplacian Matrix
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

- We normalize by replacing the a_{ij} entry with $\frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}$.

Laplacian Matrix

At any level n the graph Laplacian is a block matrix:

$$\begin{bmatrix} A & B \\ B^t & D \end{bmatrix}$$

- The A block corresponds to vertices from the $(n-1)^{\text{th}}$ level.
- The D block corresponds to new vertices introduced at the n^{th} level.
- B and B^t correspond to connections between levels $n-1$ and n.
- $4^n \Delta_n$ converges to an operator that is the correct replacement for the usual Euclidean Laplacian.

Spectral Decimation on the Laplacian

- It is known that we can relate the spectrum of Δ_n to Δ_{n-1} via the Schur complement.

$$(\Delta_n - \lambda)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A-\lambda & B \\ B^t & D-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- If λ is not an eigenvalue of D then

$$S_\lambda = A - \lambda - B(D-\lambda)^{-1}B^t = 0$$

.

Spectal Decimation on the Laplacian

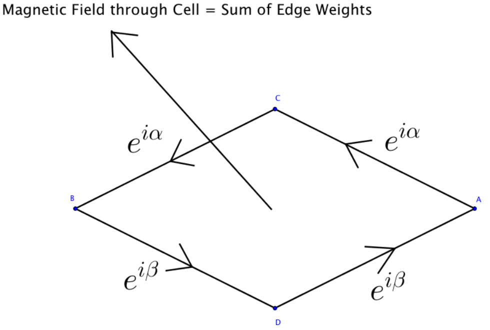
Theorem

For the Diamond Fractal there exist rational functions $\varphi_0(\lambda)$ *and* $\varphi_1(\lambda)$ *such that* $S_\lambda = \varphi_0(\lambda)\Delta_{n-1} - \varphi_1(\lambda)I$.

Corollary

If λ *is not an eigenvalue of* D *and* $\varphi_0(\lambda) \neq 0$ *then* λ *is an eigenvalue of* Δ_n *if and only if* $R(\lambda) = \frac{\varphi_1(\lambda)}{\varphi_0(\lambda)}$ *is an eigenvalue of* Δ_{n-1}.

Magnetic Laplacian



- Approximating graph becomes a weighted, directed graph.
- Edges are weighted by $e^{i\theta}$ in the direction of the edge $e^{-i\theta}$ in the opposite direction.
- $M_n u(x) = \sum_{y \backsim x} \left(u(x) - e^{i\theta_{xy}} u(y)\right)$

Spectral Decimation on Magnetic Laplacian

- A variant of Spectral Decimation still works for M_n on the Diamond Fractal.
- Instead of fixed functions φ_0 and φ_1 we have functions that depend on the magnetic field strength.

Theorem

For the Diamond Fractal there exist rational functions $\varphi_0(\lambda, \gamma)$ and $\varphi_1(\lambda, \gamma)$ such that $S_{\lambda,\gamma} = \varphi_0(\lambda, \gamma) M_{n-1} - \varphi_1(\lambda, \gamma) I$.

Spectral Decimation on Magnetic Laplacian

Example on level 1:

- $M_1 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$
- $A = D = (1-\lambda)I$
- $B = \begin{bmatrix} \frac{-e^{i\gamma}}{2} & \frac{-e^{-i\gamma}}{2} \\ \frac{-e^{-i\gamma}}{2} & \frac{-e^{i\gamma}}{2} \end{bmatrix}$

$$S_\lambda = \begin{bmatrix} 1-\lambda-\frac{1}{2(1-\lambda)} & \frac{-\cos(2\gamma)}{2(1-\lambda)} \\ \frac{-\cos(2\gamma)}{2(1-\lambda)} & 1-\lambda-\frac{1}{2(1-\lambda)} \end{bmatrix}$$

$$= \frac{\cos(2\gamma)}{2(1-\lambda)} M_0 - \left(\frac{-2\lambda^2+4\lambda-1+\cos(2\gamma)}{2(1-\lambda)} \right) I$$

Spectral Decimation on Magnetic Laplacian

- It does not immediately follow that this method can be used to reduce M_n to M_{n-1}
- "Gluing" M_n together in the right way results in the spectral decimation operators
- The additional feature we need is that the operator on each piece may be gauge-transformed.

Gauge Transforms from Level to Level

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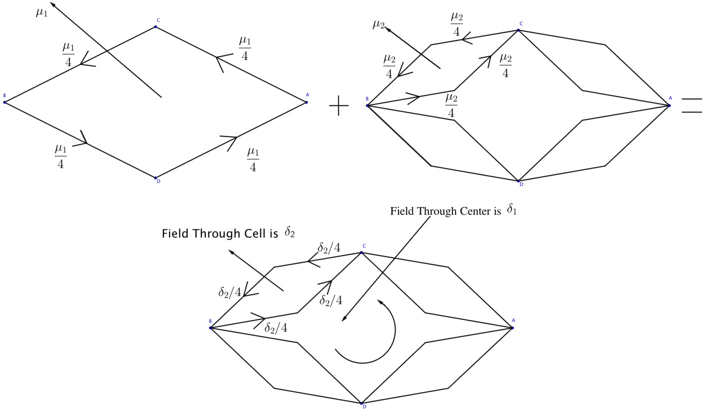
- The spectral similarity relations for the magnetic operators of the Diamond fractal can be understood through gauge transforms.
- The gauge transforms can be found through analyzing the magnetic field strength through each cell.
- The magnetic field through an approximating graph can be written as an equivalent sequence of magnetic fields.
- The equivalent magnetic field strengths determine the gauge transforms.

Gauge Transforms-Level 2 Example



$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

Gauge Transforms from Level to Level

- In General:

$$\begin{bmatrix} 1 & -2 & -4 & -16 & \dots \\ & 1 & -2 & -4 & \dots \\ & & 1 & -2 & \dots \\ & & & 1 & \dots \\ & & & & 1 \\ & & & & \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \dots \\ \mu_k \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \dots \\ \delta_k \end{bmatrix}$$

- We can invert the left most matrix in order to solve for the vector of μ values, if we know the δ field strengths from level to level.

Gauge Transforms from Level to Level

Lemma

Fix a level n. Given a magnetic field in which the field strength through a hole depends only on the level of the hole, let δ_j denote the field strength through each j^{th} level hole, and define

$$\mu_j = \delta_j + \sum_{i=j+1}^{n} 2^{2i-(2j+1)}\delta_i.$$

Define a field of strength μ_j on each j-level cell, and extend it to act as a gauge transform on all smaller cells. Then the original field is the sum of the μ_j fields.

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Gauge Transforms from Level to Level

Theorem

Suppose $\{\lambda_n\}$ is a sequence such that for each n, $\lambda_n \neq 1$ and $\mu_n \notin \frac{\pi}{2}\mathbb{Z}$. Then for each n, λ_n is an eigenvalue of M_n if and only if $R(\lambda_n, \mu_n)$ is an eigenvalue of M_{n-1}, where

$$R(\lambda, \mu) = \frac{4\lambda - 2\lambda^2 - 1}{\cos(\mu/2)} + 1.$$

Guage Transforms from Level to Level

- The exceptional case $\lambda_n = 1$ does correspond to eigenvalues, and we can calculate their multiplicity.
- Another exceptional case occurs when $\varphi_0(\lambda) = 0$.
- This does not immediately lend to eigenvalues.

Gauge Transforms from Level to Level

- This theorem gives an algorithm for finding eigenvalues of the Laplacian.
- For a given magnetic field defined by the sequence $\{\delta_i\}_{i=1}^n$, compute $\{\mu_i\}_{i=1}^n$.

Then for each i:

1. For every eigenvalue $\lambda_k^{(i-1)}$ of M_{i-1}, find its two preimages under $R(\cdot, \mu_i)$.
2. Incorporate $\frac{4^i - 4}{3}$ copies of the exceptional eigenvalue $\lambda = 1$.
3. Re-scale all eigenvalues by 4^n in order to maintain compatible energy levels between operators.
4. Take the limit as $n \to \infty$.

Gauge Transforms from Level to Level

- δ_j (the strength of the magnetic field through the level k hole) could be any sequence of fields we want.
- As a special case we consider δ_j to be proportional to area of the j^{th} level cell.
- For a suitable embedding of the fractal, the area occupied by the j^{th} level cells may be taken to be geometrically decreasing.
- $\delta_j = 4^{1-j} A^j$ for some $A < 1$.
- $\mu_j = 2^{1-2j} \left(A^{n+1} + A^{j+1} - A^j \right)$.

Predicting Eigenvalues Example

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- $R^{-1}(\lambda, \mu) = 1 \pm \sqrt{\frac{\cos(\frac{\mu}{2}) - \lambda \cos(\frac{\mu}{2}) + 1}{2}}$
- Recall the Laplacian for Level 0: $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
- Eigenvalues $= \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}$
- $R^{-1}(\begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}, \mu_1)$ gives the eigenvalues of the Laplacian of the Level 2 graph approximation

Analysis of Eigenvalues

- To analyze the spectrum of the eigenvalues, we are going to define a counting function.
- $N(x, \delta_k) =$ Number of Eigenvalues $< x$
- All of our Eigenvalues between 0 and 2 because we use the normalized Laplacian.

Spectrum of Eigenvalues

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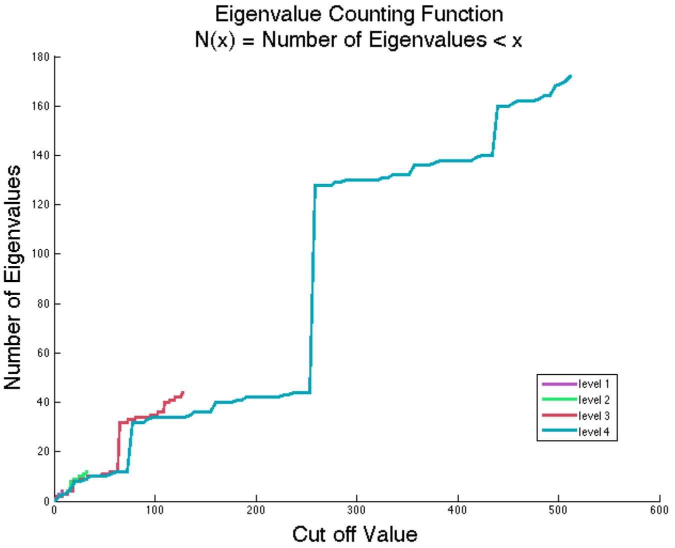

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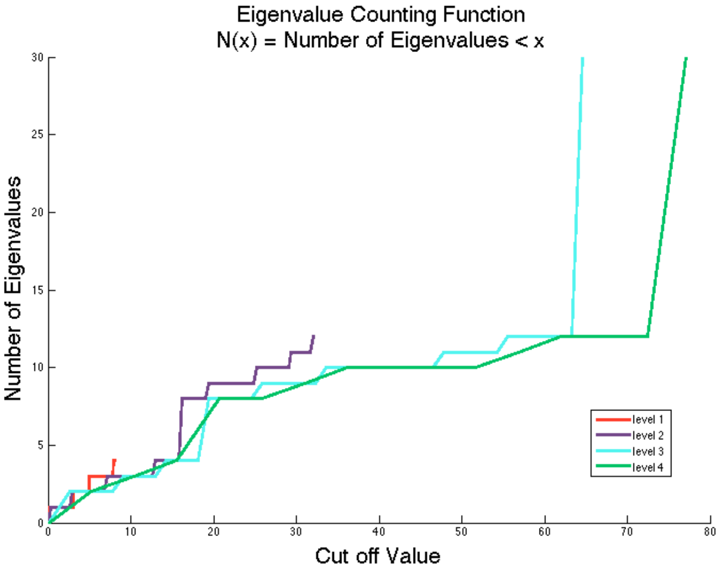
Spectrum of Eigenvalues



Strengthening Magnetic Field–Periodic Fluctuations

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VIDEO

References and Acknowledgements

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- J. Bellissard, Renormalization Group Analysis and Quasicrystals. Ideas and Methods in Analysis, Stochastics, and Applications (1992), 119–129.

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