Magnetic Spectral Decimation on the Diamond Fractal

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Motivation

Understanding the spectrum allows us to analyze the energy levels of a charged particle confined to the diamond fractal and under the influence of a magnetic field.
The Diamond Fractal

Each edge in the $n^{\text{th}}$ level approximating graph becomes a diamond in the $(n + 1)^{\text{th}}$ level.

We can represent this operation using an iterated function system of four contraction maps $f_i$.

The diamond fractal is the unique, non-empty, compact set invariant under the IFS: $K = \bigcup f_i(K)$. 
The graph Laplacian at level $n$ is an operator on functions given by

$$\Delta_n u(x) = \sum_{y \sim x} (u(x) - u(y))$$

This can be represented as the difference of the graph’s degree matrix and adjacency matrix. At level zero we have:

<table>
<thead>
<tr>
<th>Degree Matrix</th>
<th>Adjacency Matrix</th>
<th>Laplacian Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 &amp; 1 \ 1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; -1 \ -1 &amp; 1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

We normalize by replacing the $a_{ij}$ entry with $\frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}$.
At any level $n$ the graph Laplacian is a block matrix:

$$
\begin{bmatrix}
A & B \\
B^t & D
\end{bmatrix}
$$

- The $A$ block corresponds to vertices from the $(n - 1)^{th}$ level.
- The $D$ block corresponds to new vertices introduced at the $n^{th}$ level.
- $B$ and $B^t$ correspond to connections between levels $n - 1$ and $n$.
- $4^n \Delta_n$ converges to an operator that is the correct replacement for the usual Euclidean Laplacian.
It is known that we can relate the spectrum of $\Delta_n$ to $\Delta_{n-1}$ via the Schur complement.

$$(\Delta_n - \lambda)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A - \lambda & B \\ B^t & D - \lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\lambda$ is not an eigenvalue of $D$ then

$$S_{\lambda} = A - \lambda - B(D - \lambda)^{-1}B^t = 0$$
Spectral Decimation on the Laplacian

**Theorem**

For the Diamond Fractal there exist rational functions $\varphi_0(\lambda)$ and $\varphi_1(\lambda)$ such that $S_\lambda = \varphi_0(\lambda)\Delta_{n-1} - \varphi_1(\lambda)I$.

**Corollary**

If $\lambda$ is not an eigenvalue of $D$ and $\varphi_0(\lambda) \neq 0$ then $\lambda$ is an eigenvalue of $\Delta_n$ if and only if $R(\lambda) = \frac{\varphi_1(\lambda)}{\varphi_0(\lambda)}$ is an eigenvalue of $\Delta_{n-1}$. 
Magnetic Laplacian

- Approximating graph becomes a weighted, directed graph.
- Edges are weighted by $e^{i\theta}$ in the direction of the edge $e^{-i\theta}$ in the opposite direction.
- $M_n u(x) = \sum_{y \sim x} (u(x) - e^{i\theta_{xy}} u(y))$
A variant of Spectral Decimation still works for $M_n$ on the Diamond Fractal.

Instead of fixed functions $\varphi_0$ and $\varphi_1$ we have functions that depend on the magnetic field strength.

**Theorem**

For the Diamond Fractal there exist rational functions $\varphi_0(\lambda, \gamma)$ and $\varphi_1(\lambda, \gamma)$ such that

\[ S_{\lambda, \gamma} = \varphi_0(\lambda, \gamma)M_{n-1} - \varphi_1(\lambda, \gamma)I. \]
Example on level 1:

- $M_1 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$
- $A = D = (1 - \lambda)I$
- $B = \begin{bmatrix} -e^{i\gamma} & -e^{-i\gamma} \\ \frac{-e^{i\gamma}}{2} & \frac{e^{-i\gamma}}{2} \\ -e^{-i\gamma} & -e^{i\gamma} \\ \frac{-e^{-i\gamma}}{2} & \frac{e^{i\gamma}}{2} \end{bmatrix}$

\[
S_\lambda = \begin{bmatrix} 1 - \lambda - \frac{1}{2(1-\lambda)} & -\frac{\cos(2\gamma)}{2(1-\lambda)} \\ -\frac{\cos(2\gamma)}{2(1-\lambda)} & 1 - \lambda - \frac{1}{2(1-\lambda)} \end{bmatrix}
\]

\[
= \frac{\cos(2\gamma)}{2(1-\lambda)} M_0 - \left( \frac{-2\lambda^2 + 4\lambda - 1 + \cos(2\gamma)}{2(1-\lambda)} \right) I
\]
It does not immediately follow that this method can be used to reduce $M_n$ to $M_{n-1}$.

"Gluing" $M_n$ together in the right way results in the spectral decimation operators.

The additional feature we need is that the operator on each piece may be gauge-transformed.
Gauge Transforms from Level to Level

- The spectral similarity relations for the magnetic operators of the Diamond fractal can be understood through gauge transforms.
- The gauge transforms can be found through analyzing the magnetic field strength through each cell.
- The magnetic field through an approximating graph can be written as an equivalent sequence of magnetic fields.
- The equivalent magnetic field strengths determine the gauge transforms.
Gauge Transforms-Level 2 Example

\[
\begin{bmatrix}
1 & -2 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\end{bmatrix}
\]
Gauge Transforms from Level to Level

- **In General:**

\[
\begin{bmatrix}
1 & -2 & -4 & -16 & \ldots \\
1 & -2 & -4 & \ldots \\
1 & -2 & \ldots \\
1 & \ldots \\
1 & \ldots \\
\end{bmatrix}
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\mu_k \\
\end{bmatrix}
=
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\delta_k \\
\end{bmatrix}
\]

- We can invert the left most matrix in order to solve for the vector of \( \mu \) values, if we know the \( \delta \) field strengths from level to level.
Fix a level $n$. Given a magnetic field in which the field strength through a hole depends only on the level of the hole, let $\delta_j$ denote the field strength through each $j^{th}$ level hole, and define

$$\mu_j = \delta_j + \sum_{i=j+1}^{n} 2^{2i-(2j+1)} \delta_i.$$ 

Define a field of strength $\mu_j$ on each $j$-level cell, and extend it to act as a gauge transform on all smaller cells. Then the original field is the sum of the $\mu_j$ fields.
Theorem

Suppose \( \{\lambda_n\} \) is a sequence such that for each \( n \), \( \lambda_n \neq 1 \) and \( \mu_n \notin \frac{\pi}{2} \mathbb{Z} \). Then for each \( n \), \( \lambda_n \) is an eigenvalue of \( M_n \) if and only if \( R(\lambda_n, \mu_n) \) is an eigenvalue of \( M_{n-1} \), where

\[
R(\lambda, \mu) = \frac{4\lambda - 2\lambda^2 - 1}{\cos(\mu/2)} + 1.
\]
The exceptional case \( \lambda_n = 1 \) does correspond to eigenvalues, and we can calculate their multiplicity.

Another exceptional case occurs when \( \varphi_0(\lambda) = 0 \).

This does not immediately lend to eigenvalues.
This theorem gives an algorithm for finding eigenvalues of the Laplacian.

For a given magnetic field defined by the sequence \( \{\delta_i\}_{i=1}^n \), compute \( \{\mu_i\}_{i=1}^n \).

Then for each \( i \):

1. For every eigenvalue \( \lambda_k^{(i-1)} \) of \( M_{i-1} \), find its two preimages under \( R(\cdot, \mu_i) \).
2. Incorporate \( \frac{4i-4}{3} \) copies of the exceptional eigenvalue \( \lambda = 1 \).
3. Re-scale all eigenvalues by \( 4^n \) in order to maintain compatible energy levels between operators.
4. Take the limit as \( n \to \infty \).
Gauge Transforms from Level to Level

- $\delta_j$ (the strength of the magnetic field through the level $k$ hole) could be any sequence of fields we want.
- As a special case we consider $\delta_j$ to be proportional to area of the $j^{th}$ level cell.
- For a suitable embedding of the fractal, the area occupied by the $j^{th}$ level cells may be taken to be geometrically decreasing.
- $\delta_j = 4^{1-j}A^j$ for some $A < 1$.
- $\mu_j = 2^{1-2j}(A^{n+1} + A^{j+1} - A^j)$. 
Predicting Eigenvalues Example

\[ R^{-1}(\lambda, \mu) = 1 \pm \sqrt{\frac{\cos(\frac{\mu}{2}) - \lambda \cos(\frac{\mu}{2}) + 1}{2}} \]

Recall the Laplacian for Level 0:
\[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]

Eigenvalues = [0, 1, 1, 2]

\[ R^{-1}(\begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}, \mu_1) \] gives the eigenvalues of the Laplacian of the Level 2 graph approximation.
Analysis of Eigenvalues

To analyze the spectrum of the eigenvalues, we are going to define a counting function.

\[ N(x, \delta_k) = \text{Number of Eigenvalues} < x \]

All of our Eigenvalues between 0 and 2 because we use the normalized Laplacian.
Spectrum of Eigenvalues

Eigenvalue Counting Function

\[ N(x) = \text{Number of Eigenvalues} < x \]
Spectrum of Eigenvalues

Eigenvalue Counting Function

\[ N(x) = \text{Number of Eigenvalues} < x \]
Strengthening Magnetic Field–Periodic Fluctuations

VIDEO
References:


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