Determining the Spectrum of Laplacian on $3N$–Gaskets

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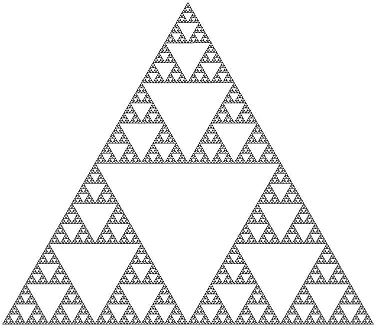
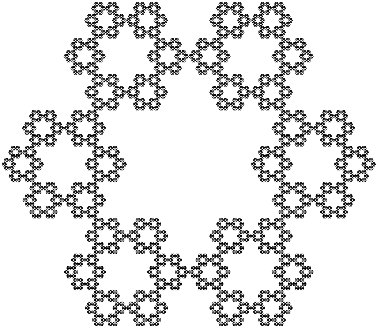
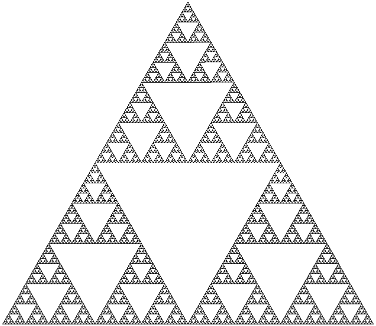
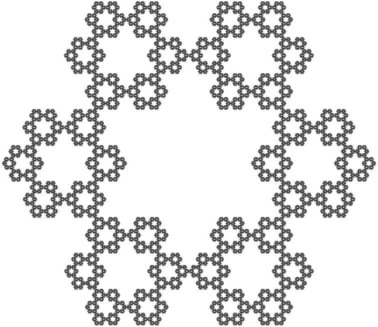




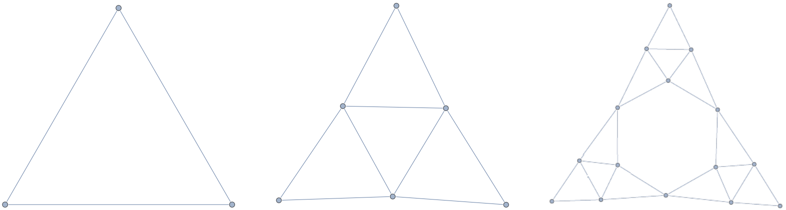
- By considering a specific fractal, we hope to further understanding of the Laplacian on fractals in general.



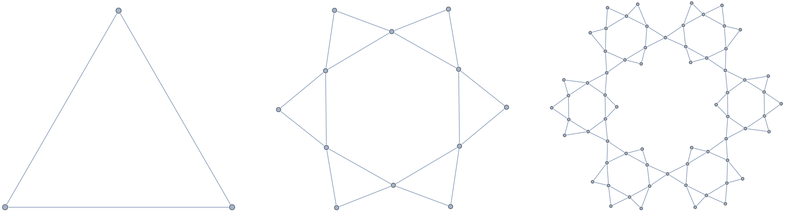


- This will lead to a greater understanding of PDEs on fractals—particularly of the heat equation and Brownian motion
- Our approach is to study the spectrum of the Laplacian on a fractal through a process called spectral decimation

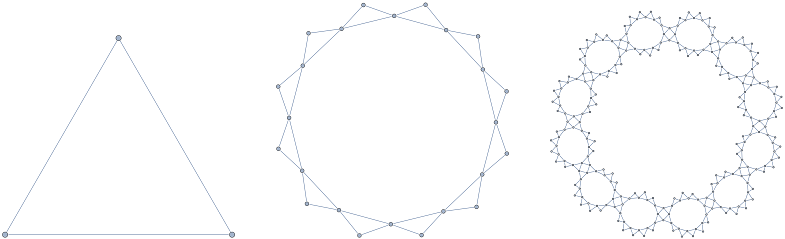
- The specific class of fractals we consider are the $3N$-Gaskets
- The $3N$-Gaskets are a family of fractals formed by an iterated function system of $3N$ contraction mappings
- The pervasive Sierpinski Triangle corresponds to $N = 1$

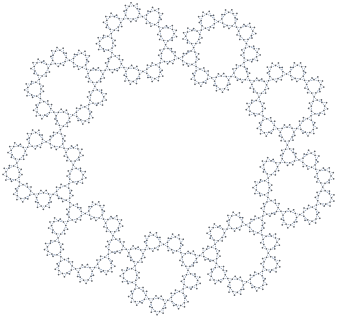


- The Hexagasket



- The 12-gasket





- We consider the graph Laplacian with point–wise definition

$$\Delta_n f(x) = f(x) - \frac{1}{\deg_n(x)} \sum_{(x,y)\in E(G_n)} f(y)$$

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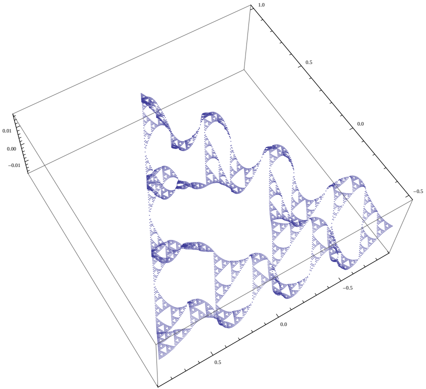
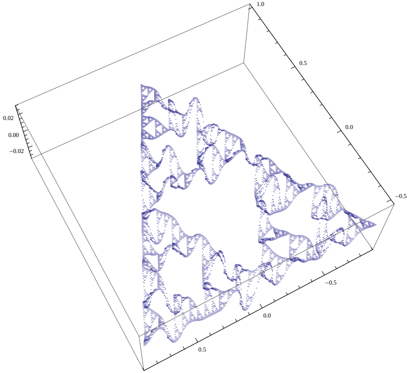
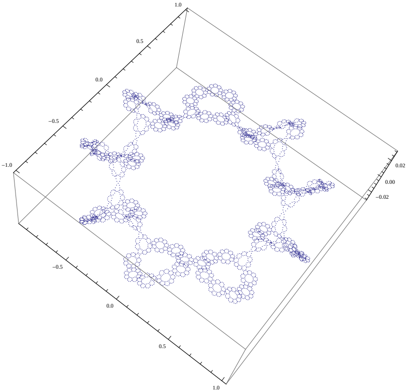
Introduction
Eigenfunction Plots

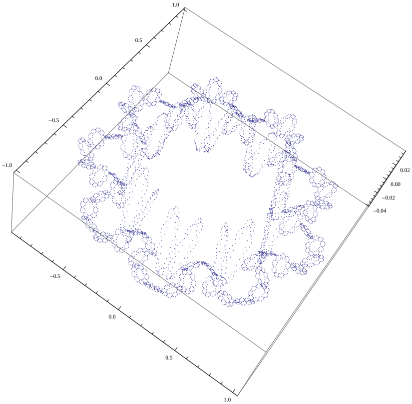
- We focus on eigenfunctions: functions f such that $\Delta f(x) = zf(x)$ for all vertices x on our graph.
- Using this information, we can find eigenfunctions of a given graph approximation.





- We refer to the set of eigenvalues of a graph G as its spectrum and write $\sigma(G)$





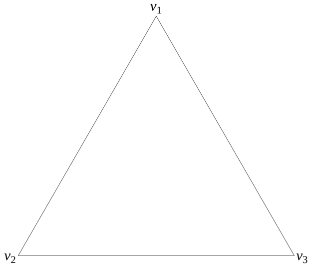
- Spectral decimation is the process of finding the eigenvalues of the graph Laplacian on a given level of the fractal from the eigenvalues on the next level
- The following items are required for spectral decimation:
    1. $R(z)$—"The Spectral Decimation Function"
    2. $\sigma(M_0)$ and $\sigma(M_1)$—sets of eigenvalues corresponding to Δ_0 and Δ_1 respectively
    3. $\phi(z)$ and $\sigma(D)$—a rational function and a set of eigenvalues of Δ_n corresponding to Dirichlet boundary conditions used to find an exceptional set of values, $E(M_1, M_0)$

Spectral Decimation
$\sigma(M_0)$

- Simply through assembling the matrix of the Laplacian for the boundary vertices,

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

we have $\sigma(M_0) = \{0, \frac{3}{2}\}$



The Spectrum

Description of Method cont.

- Ideally, by applying R^{-1} n times to the values in $\sigma(M_0)$ we would hope the entire spectrum of the graph Laplacian on any level n of the fractal, $\sigma(M_n)$, would be generated
- However, it's not so simple
  1. Some values generated this way are not eigenvalues
  2. Some eigenvalues cannot be generated this way
- It turns out we must also apply R^{-1} to the values in $\sigma(M_1)$ and $E(M_1, M_0)$

The Spectrum

Description of Method cont.

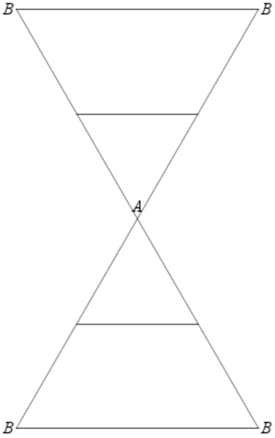
- Thus, we compute the multiplicities of these values using work from a previous UConn REU and show that the sum of the multiplicities equals the dimension of the eigenspace—the geometric multiplicity
- Finally, with a limiting argument, we obtain the spectrum of the Laplacian on the fractal, $\sigma(\Delta)$, from $\sigma(M_n)$

- The most elusive and most important tool in spectral decimation is the “spectral decimation function” $R(z)$
- By the usual method, finding this function requires manipulating the matrix of the Laplacian on the level $n = 1$ approximation
- In our case, this matrix increases in size as N increases so we use an alternative, constructive approach

The Spectrum

Calculation of $R(z)$

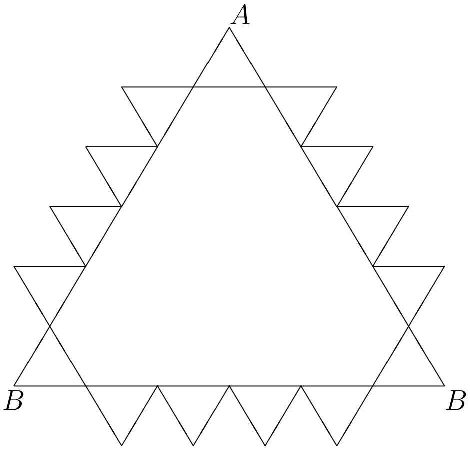
- We choose an arbitrary junction point and consider the adjacent vertices at level n, assuming an appropriate symmetry in our eigenfunctions as shown
- We wish to relate the eigenvalues on level n to those on level $n+1$ so we will need to evaluate Δ_n and Δ_{n+1}
- For the latter, we need values for the unlabeled vertices



The Spectrum

Calculation of $R(z)$

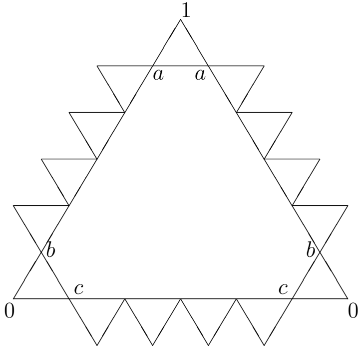
- Equivalently, we must find eigenfunctions that live on the following graph



The Spectrum

Calculation of $R(z)$

- To do this, we entertain the problem of creating eigenfunctions that live on the graph with the boundary conditions $1, 0, 0$
- Then, taking linear combinations of these eigenfunctions will yield eigenfunctions on the graph with boundary conditions A, B, B

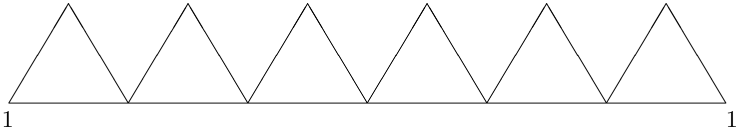


The Spectrum

Calculation of $R(z)$

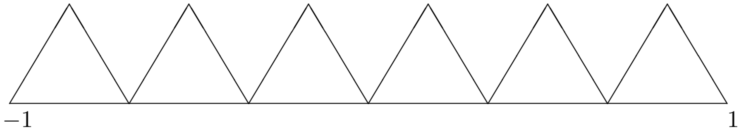
- Simplifying the problem again, we find an eigenbasis for one of the three sides of the graph containing two eigenfunctions
- A symmetric eigenfunction

$$f_1(k) = \frac{\cos((k - \frac{N-1}{2})\theta)}{\cos(\frac{N-1}{2}\theta)} = \frac{\cos((k - \frac{N-1}{2})\arccos(1-2z))}{\cos(\frac{N-1}{2}\arccos(1-2z))}$$

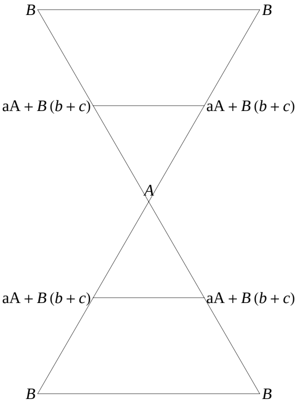


- An anti-symmetric eigenfunction:

$$f_2(k) = -\frac{\sin((k - \frac{N-1}{2})\theta)}{\sin(\frac{N-1}{2}\theta)} = -\frac{\sin((k - \frac{N-1}{2})\arccos(1-2z))}{\sin(\frac{N-1}{2}\arccos(1-2z))}$$



- These eigenfunctions on the edges can be glued together to give eigenfunctions on the full graph with boundary values $1, 0, 0$
- Thus, we have eigenfunctions for the graph with boundary values A, B, B by taking linear combinations



- We evaluate the Laplacian at the vertex with value A on the two consecutive levels n and $n+1$ and solve for z_{n+1}:

$$R(z_{n+1}) = z_n = \frac{a+b+c+z_{n+1}-1}{b+c}$$

- Our eigenfunctions provide values for a, b, and c in terms of z_{n+1}

- Thus, we have the following form for $R(z)$:
- If N is even then

$$R(z) = \frac{(z-1)\sqrt{z}U_{N-1}\left(\sqrt{z}\right)\left(2T_N(1-2z)+2U_{N-1}(1-2z)+1\right)}{T_N\left(\sqrt{z}\right)}$$

- If N is odd then

$$R(z) = \frac{\sqrt{z}T_N\left(\sqrt{z}\right)\left(2T_N(1-2z)+2U_{N-1}(1-2z)+1\right)}{U_{N-1}\left(\sqrt{z}\right)}$$

- The spectral decimation function is rational
- $T_N(z)$ and $U_N(z)$ are Chebyshev polynomials of the first and second kinds respectively

- The poles of $R(z)$ for N even are

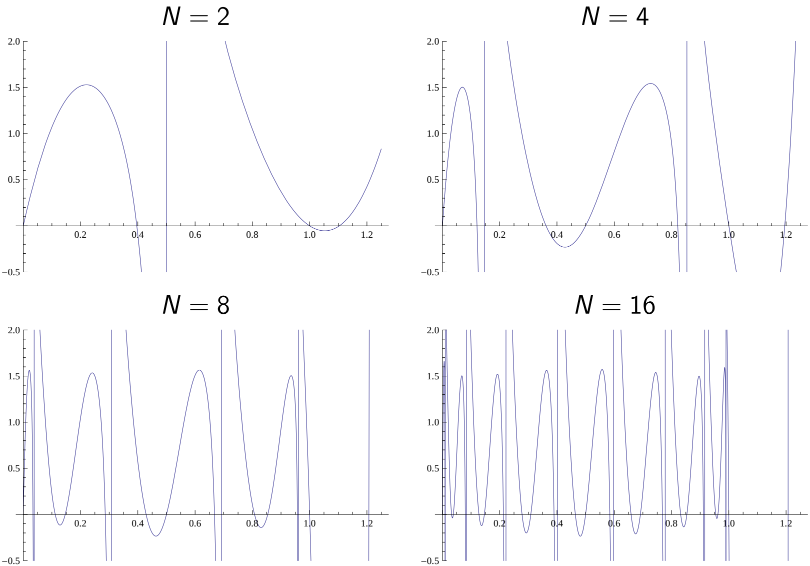
$$\zeta_k = \{\cos^2\left(\frac{(m-\frac{1}{2})\pi}{N}\right) : m = 0, ..., N-1\}$$

- and for N odd,

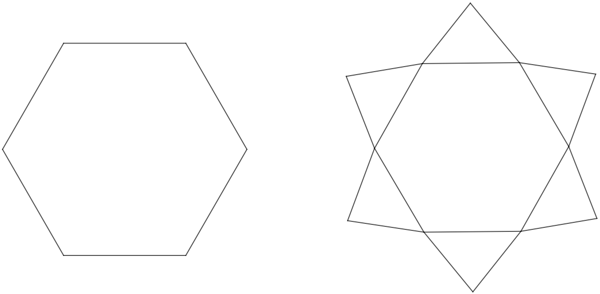
$$\zeta_k = \{\cos^2\left(\frac{m\pi}{N}\right) : m = 0, 1..., N-1\}$$

The Spectrum

$R(z)$ Plots–N Even



- Consider the following graph and its extension:

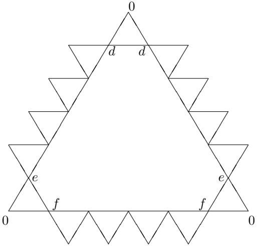


- $\sigma(M_1) = \{\frac{3}{2}\} \bigcup \{\frac{1-\cos(\theta)}{2} : \theta = \frac{2m\pi}{3N}$ and $m = 0, 1, ..., 3N-1\}$

The Spectrum

$\sigma(D)$

- We consider graphs with Dirichlet boundary conditions:



- Symmetric Case:

$$zd = d - \frac{1}{4}(d + el + dr),\ ze = e - \frac{1}{4}(f + el + dr),\ zf = f - \frac{1}{4}(e + fl + fr)$$

- Skew Symmetric Case:

$$zd = d - \frac{1}{4}(-d + el + dr),\ ze = e - \frac{1}{4}(f + el + dr),\ zf = f - \frac{1}{4}(e - fl + fr)$$

- If N is even then

$$\phi(z) = \frac{(3-2z)T_N\left(\sqrt{z}\right)}{\left(T_N\left(\sqrt{z}\right) - 2(z-1)\sqrt{z}U_{N-1}\left(\sqrt{z}\right)\right)\left(2T_N(1-2z) + 2U_{N-1}(1-2z) + 1\right)}$$

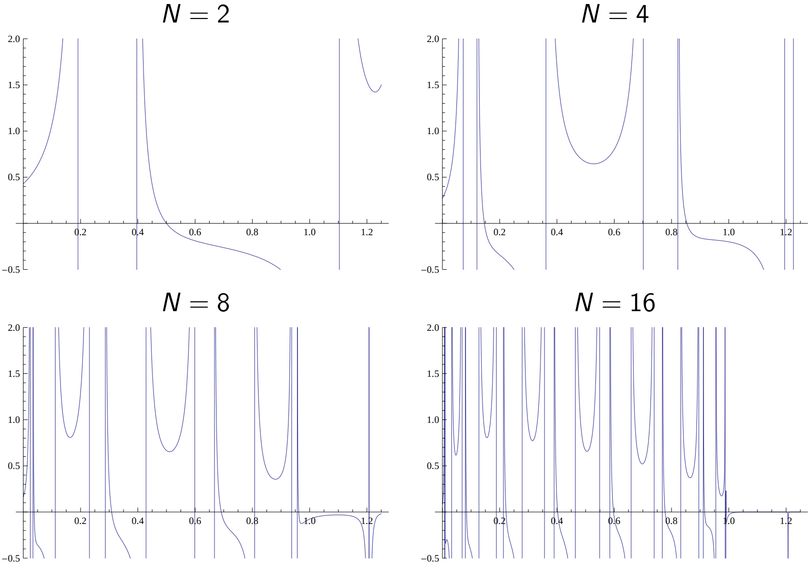
- If N is odd then

$$\phi(z) = \frac{(3-2z)U_{N-1}\left(\sqrt{z}\right)}{\left(U_{N-1}\left(\sqrt{z}\right) - 2\sqrt{z}T_N\left(\sqrt{z}\right)\right)\left(2T_N(1-2z) + 2U_{N-1}(1-2z) + 1\right)}$$

- Like $R(z)$, $\phi(z)$ is rational

The Spectrum

$\phi(z)$ plots–N even



- For N even,

$$\begin{aligned}\sigma(M_n) &= \left\{0, \frac{3}{2}\right\} \cup \bigcup_{m=0}^{n-1} R_{-m}\Big(\sigma(M) - \{1\}\Big) \\ &\cup \left(\bigcup_{m=0}^{n-2} R_{-m}\Big(\sigma(D) - \big(\, \{z : \phi(z) = 0\} \right. \\ &\cup \left. \{z : T_N(\sqrt{z}) - 2(z-1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0\}\,\big)\Big)\right)\end{aligned}$$

- For N odd,

$$\begin{aligned}\sigma(M_n) &= \left\{0, \frac{3}{2}\right\} \cup \bigcup_{m=0}^{n-1} R_{-m}\Big(\sigma(M)\Big) \\ &\cup \left(\bigcup_{m=0}^{n-2} R_{-m}\Big(\sigma(D) - \big(\, \{z : \phi(z) = 0\} \right. \\ &\cup \left. \{z : U_{N-1}(\sqrt{z}) - 2\sqrt{z}T_N(\sqrt{z}) = 0\}\,\big)\Big)\right)\end{aligned}$$

The Spectrum

Main Results cont.

- For N even,

$$\begin{aligned}\sigma(\Delta) &= \left\{\frac{3}{2}\right\} \cup \Bigg(\bigcup_{m=0}^{\infty} R_{-m}\Big(\Big\{\sin^2(\frac{m\pi}{3N}) : k = 0, \ldots, 3N-1\Big\} \\ &\cup \quad \left\{z : T_N(\sqrt{z}) - 2(z-1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0\right\} \\ &\cup \quad \left\{\cos^2\left(\frac{m\pi}{N}\right) : m = 1, \ldots, \frac{N}{2}\right\}\Big)\Bigg)\end{aligned}$$

- For N odd,

$$\begin{aligned}\sigma(\Delta) &= \left\{\frac{3}{2}\right\} \cup \Bigg(\bigcup_{m=0}^{\infty} R_{-m}\Big(\Big\{\sin^2(\frac{m\pi}{3N}) : k = 0, \ldots, 3N-1\Big\} \\ &\cup \quad \{z : 2T_N(1-2z) + 2U_{N-1}(1-2z) + 1 = 0\}) \\ &\cup \quad \left\{\cos^2\left(\frac{\left(m - \frac{1}{2}\right)\pi}{N}\right) : m = 1, \ldots, \frac{N-1}{2}\right\}\Big)\Bigg)\end{aligned}$$