Determining the Spectrum of Laplacian on 3N–Gaskets

N. Gupta  D. Kelleher  M. Margenot  J. Marsh  W. Oakley  A. Teplyaev

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By considering a specific fractal, we hope to further understanding of the Laplacian on fractals in general.
This will lead to a greater understanding of PDEs on fractals—particularly of the heat equation and Brownian motion.

Our approach is to study the spectrum of the Laplacian on a fractal through a process called spectral decimation.
Introduction
The $3N$-Gaskets

- The specific class of fractals we consider are the $3N$-Gaskets.
- The $3N$-Gaskets are a family of fractals formed by an iterated function system of $3N$ contraction mappings.
- The pervasive Sierpinski Triangle corresponds to $N = 1$. 

![Diagram of $3N$-Gaskets](image)
Introduction

The 3N-Gaskets cont.

▶ The Hexagasket

▶ The 12-gasket
We consider the graph Laplacian with point-wise definition

$$\Delta_n f(x) = f(x) - \frac{1}{\text{deg}_n(x)} \sum_{(x,y) \in E(G_n)} f(y)$$
Introduction

Eigenfunction Plots

- We focus on eigenfunctions: functions $f$ such that $\Delta f(x) = zf(x)$ for all vertices $x$ on our graph.
- Using this information, we can find eigenfunctions of a given graph approximation.
We refer to the set of eigenvalues of a graph $G$ as its spectrum and write $\sigma(G)$.
The Spectrum
Description of Method

- Spectral decimation is the process of finding the eigenvalues of the graph Laplacian on a given level of the fractal from the eigenvalues on the next level.

- The following items are required for spectral decimation:
  1. \( R(z) \)—“The Spectral Decimation Function”
  2. \( \sigma(M_0) \) and \( \sigma(M_1) \)—sets of eigenvalues corresponding to \( \Delta_0 \) and \( \Delta_1 \) respectively
  3. \( \phi(z) \) and \( \sigma(D) \)—a rational function and a set of eigenvalues of \( \Delta_n \) corresponding to Dirichlet boundary conditions used to find an exceptional set of values, \( E(M_1, M_0) \)
Simply through assembling the matrix of the Laplacian for the boundary vertices,

\[
\begin{pmatrix}
1 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & 1 & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & 1
\end{pmatrix}
\]

we have \( \sigma(M_0) = \{0, \frac{3}{2}\} \)
Ideally, by applying $R^{-1}$ $n$ times to the values in $\sigma(M_0)$ we would hope the entire spectrum of the graph Laplacian on any level $n$ of the fractal, $\sigma(M_n)$, would be generated.

However, it's not so simple:

1. Some values generated this way are not eigenvalues
2. Some eigenvalues cannot be generated this way

It turns out we must also apply $R^{-1}$ to the values in $\sigma(M_1)$ and $E(M_1, M_0)$.
Thus, we compute the multiplicities of these values using work from a previous UConn REU and show that the sum of the multiplicities equals the dimension of the eigenspace—the geometric multiplicity.

Finally, with a limiting argument, we obtain the spectrum of the Laplacian on the fractal, $\sigma(\Delta)$, from $\sigma(M_n)$. 
The most elusive and most important tool in spectral decimation is the “spectral decimation function” \( R(z) \).

By the usual method, finding this function requires manipulating the matrix of the Laplacian on the level \( n = 1 \) approximation.

In our case, this matrix increases in size as \( N \) increases so we use an alternative, constructive approach.
The Spectrum
Calculation of $R(z)$

- We choose an arbitrary junction point and consider the adjacent vertices at level $n$, assuming an appropriate symmetry in our eigenfunctions as shown.
- We wish to relate the eigenvalues on level $n$ to those on level $n + 1$ so we will need to evaluate $\Delta_n$ and $\Delta_{n+1}$.
- For the latter, we need values for the unlabeled vertices.
Equivalently, we must find eigenfunctions that live on the following graph.
To do this, we entertain the problem of creating eigenfunctions that live on the graph with the boundary conditions 1, 0, 0.

Then, taking linear combinations of these eigenfunctions will yield eigenfunctions on the graph with boundary conditions $A, B, B$. 
Simplifying the problem again, we find an eigenbasis for one of the three sides of the graph containing two eigenfunctions

A symmetric eigenfunction

\[ f_1(k) = \frac{\cos((k - \frac{N-1}{2})\theta)}{\cos(\frac{N-1}{2} \theta)} = \frac{\cos((k - \frac{N-1}{2}) \arccos(1 - 2z))}{\cos(\frac{N-1}{2} \arccos(1 - 2z))} \]
An anti-symmetric eigenfunction:

\[ f_2(k) = -\frac{\sin((k - \frac{N-1}{2})\theta)}{\sin(\frac{N-1}{2}\theta)} = -\frac{\sin((k - \frac{N-1}{2})\arccos(1 - 2z))}{\sin(\frac{N-1}{2}\arccos(1 - 2z))} \]
These eigenfunctions on the edges can be glued together to give eigenfunctions on the full graph with boundary values 1, 0, 0.

Thus, we have eigenfunctions for the graph with boundary values $A, B, B$ by taking linear combinations.
We evaluate the Laplacian at the vertex with value $A$ on the two consecutive levels $n$ and $n+1$ and solve for $z_{n+1}$:

$$R(z_{n+1}) = z_n = \frac{a + b + c + z_{n+1} - 1}{b + c}$$

Our eigenfunctions provide values for $a$, $b$, and $c$ in terms of $z_{n+1}$.
Thus, we have the following form for $R(z)$:

If $N$ is even then

$$R(z) = \frac{(z - 1)\sqrt{z}U_{N-1}(\sqrt{z}) (2T_N(1 - 2z) + 2U_{N-1}(1 - 2z) + 1)}{T_N(\sqrt{z})}$$

If $N$ is odd then

$$R(z) = \frac{\sqrt{z}T_N(\sqrt{z}) (2T_N(1 - 2z) + 2U_{N-1}(1 - 2z) + 1)}{U_{N-1}(\sqrt{z})}$$

The spectral decimation function is rational

$T_N(z)$ and $U_N(z)$ are Chebyshev polynomials of the first and second kinds respectively
The Spectrum
The Poles of $R(z)$

- The poles of $R(z)$ for $N$ even are
  \[ \zeta_k = \{\cos^2 \left( \frac{(m - \frac{1}{2})\pi}{N} \right) : m = 0, ..., N - 1 \} \]

- and for $N$ odd,
  \[ \zeta_k = \{\cos^2 \left( \frac{m\pi}{N} \right) : m = 0, 1 ..., N - 1 \} \]
The Spectrum

$R(z)$ Plots–$N$ Even

$N = 2$

$N = 4$

$N = 8$

$N = 16$
Consider the following graph and its extension:

\[ \sigma(M_1) = \left\{ \frac{3}{2} \right\} \cup \left\{ \frac{1 - \cos(\theta)}{2} : \theta = \frac{2m\pi}{3N} \text{ and } m = 0, 1, \ldots, 3N - 1 \right\} \]
We consider graphs with Dirichlet boundary conditions:

**Symmetric Case:**

\[ zd = d - \frac{1}{4} (d + el + dr), \quad ze = e - \frac{1}{4} (f + el + dr), \quad zf = f - \frac{1}{4} (e + fl + fr) \]

**Skew Symmetric Case:**

\[ zd = d - \frac{1}{4} (-d + el + dr), \quad ze = e - \frac{1}{4} (f + el + dr), \quad zf = f - \frac{1}{4} (e - fl + fr) \]
The Spectrum

$\phi(z)$

▶ If $N$ is even then

$$\phi(z) = \frac{(3 - 2z) T_N \left( \sqrt{z} \right)}{\left( T_N \left( \sqrt{z} \right) - 2(z - 1) \sqrt{z} U_{N-1} \left( \sqrt{z} \right) \right) \left( 2 T_N(1 - 2z) + 2 U_{N-1}(1 - 2z) + 1 \right)}$$

▶ If $N$ is odd then

$$\phi(z) = \frac{(3 - 2z) U_{N-1} \left( \sqrt{z} \right)}{\left( U_{N-1} \left( \sqrt{z} \right) - 2 \sqrt{z} T_N \left( \sqrt{z} \right) \right) \left( 2 T_N(1 - 2z) + 2 U_{N-1}(1 - 2z) + 1 \right)}$$

▶ Like $R(z)$, $\phi(z)$ is rational
The Spectrum

$\phi(z)$ plots–$N$ even

$N = 2$

$N = 4$

$N = 8$

$N = 16$
The Spectrum
Main Results

- For \(N\) even,

\[
\sigma(M_n) = \left\{ 0, \frac{3}{2} \right\} \cup \bigcup_{m=0}^{n-1} R_{-m}\left(\sigma(M) - \{1\}\right)
\]

\[
\cup \left( \bigcup_{m=0}^{n-2} R_{-m}\left(\sigma(D) - \{z : \phi(z) = 0\}\right) \right)
\]

\[
\cup \left\{ z : T_N(\sqrt{z}) - 2(z - 1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0 \right\}
\]

- For \(N\) odd,

\[
\sigma(M_n) = \left\{ 0, \frac{3}{2} \right\} \cup \bigcup_{m=0}^{n-1} R_{-m}\left(\sigma(M)\right)
\]

\[
\cup \left( \bigcup_{m=0}^{n-2} R_{-m}\left(\sigma(D) - \{z : \phi(z) = 0\}\right) \right)
\]

\[
\cup \left\{ z : U_{N-1}(\sqrt{z}) - 2\sqrt{z}T_N(\sqrt{z}) = 0 \right\}
\]
For $N$ even,

$$\sigma(\Delta) = \left\{ \frac{3}{2} \right\} \cup \left( \bigcup_{m=0}^{\infty} R_{-m} \left( \left\{ \sin^2 \left( \frac{m\pi}{3N} \right) : k = 0, \ldots, 3N - 1 \right\} \right) \right.$$ 

$$\cup \left\{ z : T_N(\sqrt{z}) - 2(z - 1)\sqrt{z} U_{N-1}(\sqrt{z}) = 0 \right\}$$ 

$$\cup \left\{ \cos^2 \left( \frac{m\pi}{N} \right) : m = 1, \ldots, \frac{N}{2} \right\} \right) \right)$$

For $N$ odd,

$$\sigma(\Delta) = \left\{ \frac{3}{2} \right\} \cup \left( \bigcup_{m=0}^{\infty} R_{-m} \left( \left\{ \sin^2 \left( \frac{m\pi}{3N} \right) : k = 0, \ldots, 3N - 1 \right\} \right) \right.$$ 

$$\cup \left\{ z : 2T_N(1 - 2z) + 2U_{N-1}(1 - 2z) + 1 = 0 \right\}$$ 

$$\cup \left\{ \cos^2 \left( \frac{(m - \frac{1}{2})\pi}{N} \right) : m = 1, \ldots, \frac{N-1}{2} \right\} \right) \right)$$