Random Walks on Barycentric Subdivisions and the Strichartz Hexacarpet

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University of Connecticut Mathematics REU 2011

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YMC at Ohio State University, August 19, 2011
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What is barycentric subdivision?

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This is what the barycentric subdivision of a tetrahedron (3-simplex) looks like
We can repeatedly barycentrically divide a simplex $K$. We denote the $n$-th barycentric subdivision as $K^n$. 
Adjacent simplices

- We say that two $d$-simplexes in $K^n$ are adjacent if they intersect along a $d - 1$ simplex.
Given $K^n$, we define a graph by assigning a vertex to each of simplices and an edge if those vertexes are adjacent.
Approximating Graphs (Cont.)

Here is the fourth level approximation:
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So we have a way of specifying vertices. How do we find the edges?

If we think of a good naming scheme, then given a word of length $n$ (ex. 0534134), we can find its neighbors. But more on this later...
- We can perform contractions. This is the same as tacking on a letter to the front of the word.
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Let the length of the words go to infinity with the same connections for each of the finite length words, we obtain the Strichartz Hexacarpet. But what does this look like?
Cayley graph of $S_n$ with adjacent elements swapped
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Permutohedron

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2nd Level approximation
On a finite graph approximation, the Laplacian, $\Delta$, is defined as

$$-\Delta_n u(x) = \sum_{x \sim y} (u(x) - u(y))$$  \hspace{1cm} (1)

Using the Laplacian, we can then determine the eigenvalues and eigenfunctions using the equation

$$-\Delta_n u(x) = \lambda u(x)$$  \hspace{1cm} (2)
An Example on Determining the Laplacian

The Laplacian equation again $-\Delta_n u(x) = \sum_{x \sim y} (u(x) - u(y))$

$$
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & -1 & -1 & -1 \\
2 & -1 & 2 & -1 & 0 \\
3 & -1 & -1 & 3 & -1 \\
4 & -1 & 0 & -1 & 2
\end{pmatrix}
$$

This is the negative matrix of the Laplacian.

So we can see, finding the neighbors of each cell is very important.
We have developed an algorithm to find the neighbors of any cell in an arbitrary $d$-simplex. We will present the method on a tetrahedron so that you see the intuition behind the algorithm.
Defining the Naming Scheme

**Definition**

Any cell, \( A \), in one Barycentric subdivision of a tetrahedron has one letter word

\[
A = (a_1, a_2, a_3, a_4)
\]

with \( \{a_1, a_2, a_3, a_4\} = \{0, 1, 2, 3\} \), each necessarily unique. We call \( \{0, 1, 2, 3\} \) *characters*

- \( a_1 \) tells us the vertex \( A \) shares with its parent cell
- \( a_1a_2 \) gives us the edge of the parent on which the second vertex is
- \( a_1a_2a_3 \) gives us the face of the parent on which the third vertex is
- \( a_4 \) is the unused character. It is the vertex opposite to the face of \( K^0 \) that \( A \) intersects
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Definition

The vertices of a cell $A$ that results from a Barycentric subdivision are labeled by looking at each vertex of $K^0$ that has is named character $a_i$ then finding the vertex of $A$ that is closest to it using the Euclidean metric. This vertex of $A$ is then named $a_i$. 

![Diagram of Barycentric subdivision]
As we wish to reference cells in further Barycentric subdivisions, $K^n$, the words become $n \times 4$ matrices where each $i$th row, $(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4})$, gives the level-$i$ cell that contains $A$.

An example of this: \[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
\end{pmatrix}
\]
An Example of Naming Higher Levels: 

$$
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3
\end{pmatrix}
$$
A cell can have at most 4 neighbors because it has 4 faces. All cells have 4 neighbors, but some have 3 neighbors if they are boundary cells which are defined as:

**Definition**

A cell is an *boundary cell* if one of its faces intersects the face of $K^0$.
Finding the First Three Neighbors

**Corollary**

*Every cell in $B^1$ is a boundary cell.*

**Proposition**

*Any cell $A \in K^n$, $n \geq 1$, has three neighbors in its $(n-1)$ parent cell.*
Finding the First Three Neighbors

Lemma

Any cell $A \in K^n$ has 3 inner neighbors in the same $(n-1)$-cell obtained by applying the transposition $\sigma_i = (a_{ni}, a_{n(i+1)})$ to the $n$th row of $A$ for $i = 1, 2, 3$.

Example: Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

Its 3 neighbors are:

$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \end{pmatrix}$,  $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix}$,  $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix}$

This transposition gives us the permutahedron.
Let us consider \((0, 1, 2, 3)\) which has neighbors \((1, 0, 2, 3), (0, 2, 1, 3), (0, 1, 3, 2)\)
**Proposition**

If \( A \not\in \partial K^n \) then \( A \) has a fourth neighbor in a \( j \) cell for some \( j \leq n - 1 \).

Suppose that \( A, B \in K^n \) and \( A \sim B \) (\( B \) is the fourth neighbor), then \( A \)'s parent is touching \( B \)'s parent.
Finding the Fourth Neighbor

Using these ideas, we can develop an algorithm to find the fourth neighbor (details omitted)

(1) Find the parent cell that neighbors $A$.
(2) Find a homomorphism that converts the characters from $A$’s coordinates to the neighbor’s coordinates
(3) Apply this homomorphism and obtain the neighbor
The heat equation is
\[ \frac{\partial u}{\partial t} - c\Delta u(x, t) = 0 \]  
where \( t > 0 \) where \( \Delta \) is a Laplacian in the space variable \( x \) and \( c \) is a positive constant.

For a level 3 subdivision of a tetrahedron we get the resistance is \( \rho = 13.4274 \).