Random Walks on Barycentric Subdivisions and the Strichartz Hexacarpet

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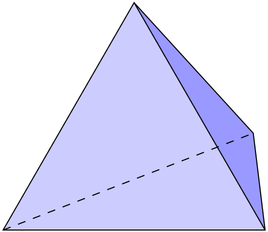
YMC at Ohio State University, August 19, 2011

Simplexes and Simplicial Complices

- A d-dimensional simplex is the convex hull of $d+1$ points in general position.



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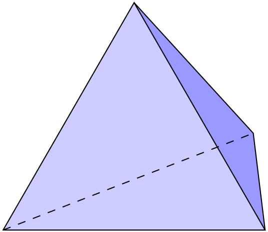

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Simplicial complexes

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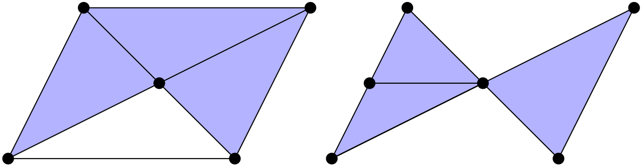

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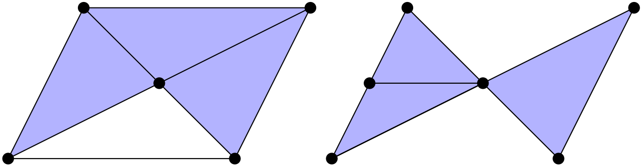

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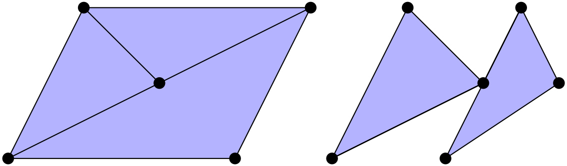

What is a barycenter?

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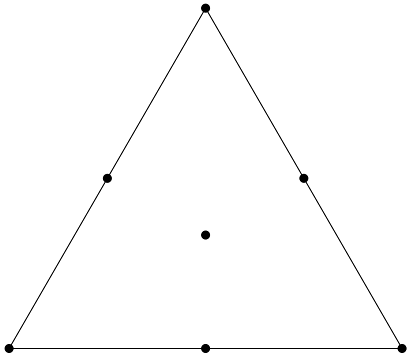

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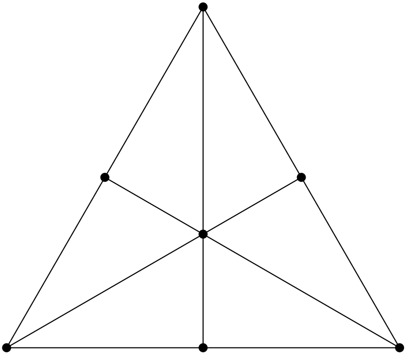

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- Intuitively, we subdivide all of the faces of a simplex and connect the subdivision of the faces to the barycenter of that simplex.

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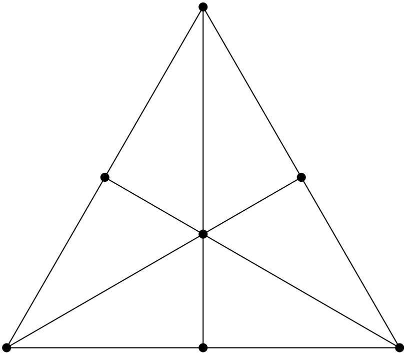

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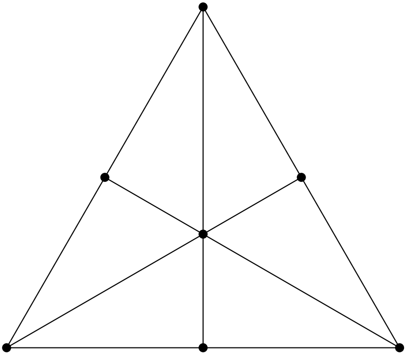

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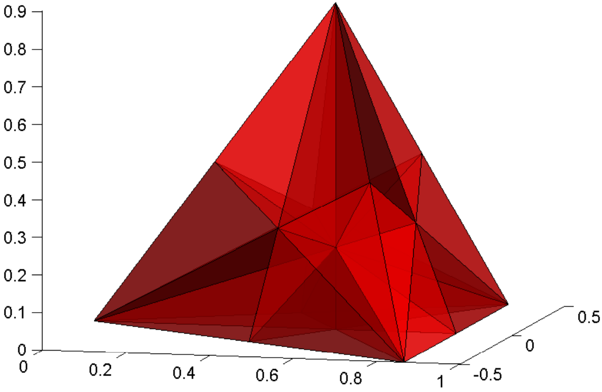
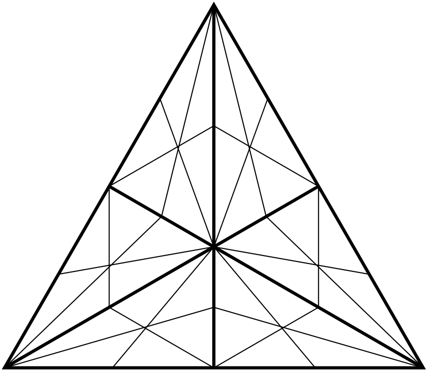
3-D Barycentric Subdivision

This is what the barycentric subdivision of a tetrahedron (3-simplex) looks like



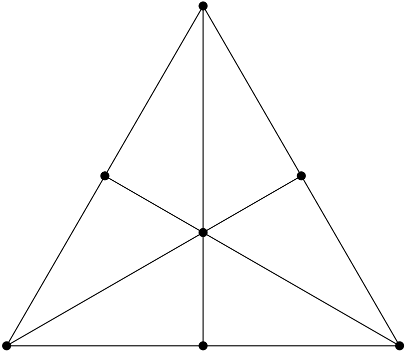
Repeated Barycentric Subdivision

- We can repeatedly barycentrically divide a simplex K. We denote the n-th barycentric subdivision as K^n.



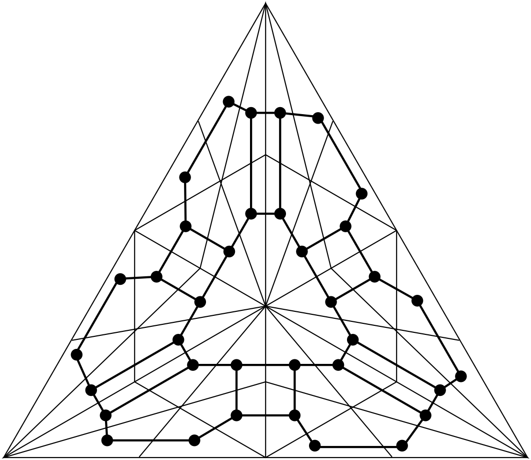
Adjacent simplices

- We say that two d-simplexes in K^n are adjacent if they intersect along a $d-1$ simplex.



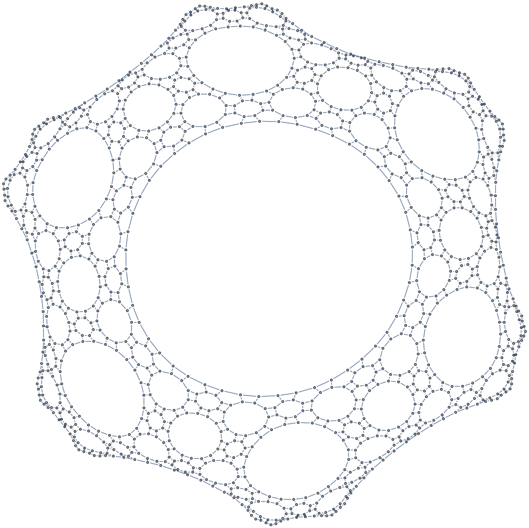
Approximating graphs

- Given K^n, we define a graph by assigning a vertex to each of simplices and an edge if those vertexes are adjacent.



Approximating Graphs (Cont.)

Here is the fourth level approximation:



How to Determine Graphs

- Number of vertices of this graphs grows as $((d+1)!)^n$ where d is the dimension of the original simplex.
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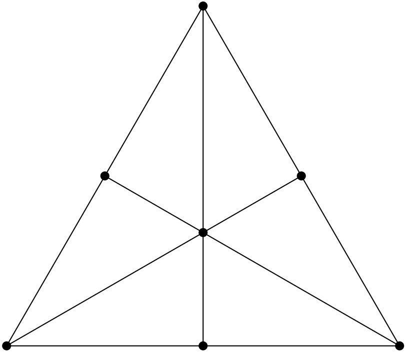
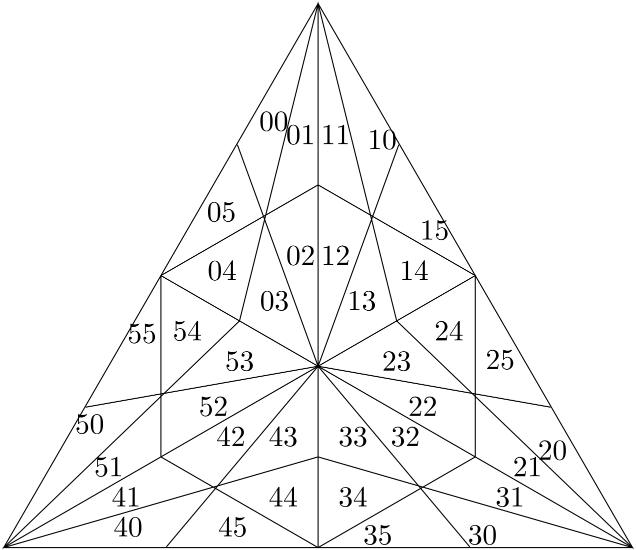


Words and Alphabets



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- So we have a way of specifying vertices. How do we find the edges?
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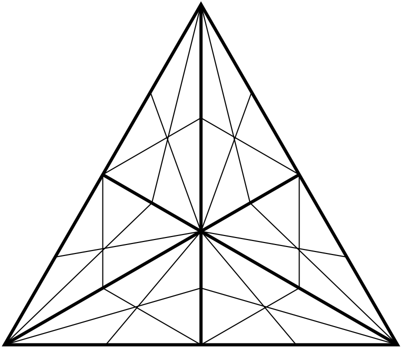
Contraction and Projection Mappings



- We can perform contractions. This is the same as tacking on a letter to the front of the word.
- We also have a projection map from a cell to it's parent cell by deleting the last letter of the word.

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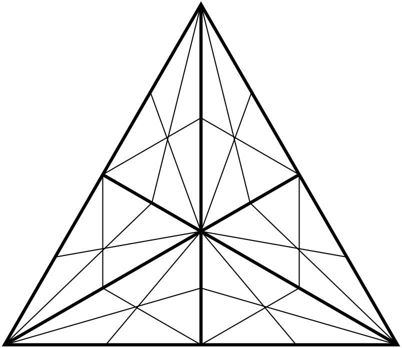

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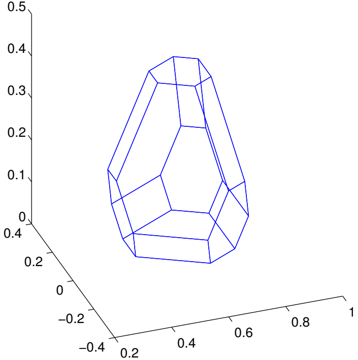
- Let the length of the words go to infinity with the same connections for each of the finite length words, we obtain the Strichartz Hexacarpet. But what does this look like?

Permutohedron



- Cayley graph of S_n with adjacent elements swapped

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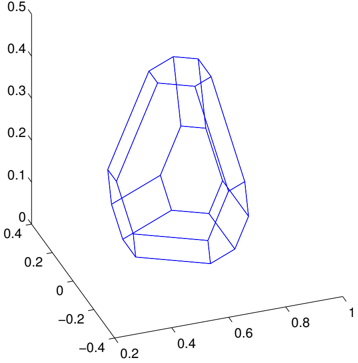

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Permutohedron

- Take all the permutations the vector $(1, 2, \ldots, n+1) \in \mathbb{R}^{n+1}$.
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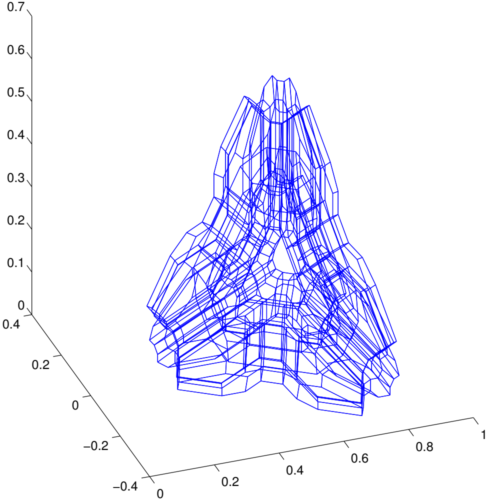
2nd Level approximation



Defining the Laplacian, Eigenvalues, and Eigenfunctions

- On a finite graph approximation, the Laplacian, Δ, is defined as

$$-\Delta_n u(x) = \sum_{x \underset{n}{\sim} y} (u(x) - u(y)) \tag{1}$$

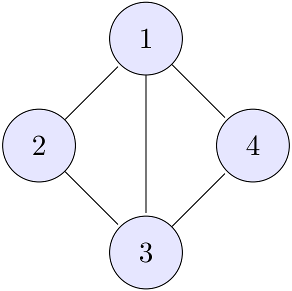
- Using the Laplacian, we can then determine the eigenvalues and eigenfunctions using the equation

$$-\Delta_n u(x) = \lambda u(x) \tag{2}$$

An Example on Determining the Laplacian

The Laplacian equation again $-\Delta_n u(x) = \sum_{x \underset{n}{\sim} y} (u(x) - u(y))$

	1	2	3	4
1	3	−1	−1	−1
2	−1	2	−1	0
3	−1	−1	3	−1
4	−1	0	−1	2



This is the negative matrix of the Laplacian.

So we can see, finding the neighbors of each cell is very important.

Working on the 3-Simplex

We have developed an algorithm to find the neighbors of any cell in a arbitrary d-simplex. We will present the method on a tetrahedron so that you see the intuition behind the algorithm.

Defining the Naming Scheme

Definition

Any cell, A, in one Barycentric subdivision of a tetrahedron has one letter *word*

$$A = (a_1, a_2, a_3, a_4)$$

with $\{a_1, a_2, a_3, a_4\} = \{0, 1, 2, 3\}$, each necessarily unique. We call $\{0, 1, 2, 3\}$ *characters*

- a_1 tells us the vertex A shares with its parent cell
- $\overline{a_1 a_2}$ gives us the edge of the parent on which the second vertex is
- $\Delta a_1 a_2 a_3$ gives us the face of the parent on which the third vertex is
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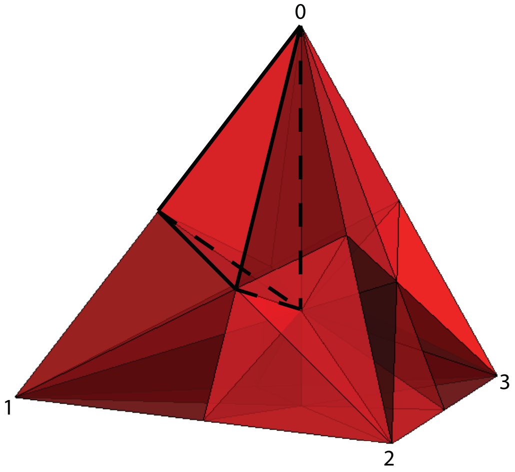
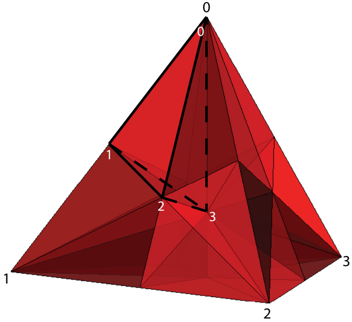
An Example of Naming a Level 1 Subdivision: $(0, 1, 2, 3)$



Naming Higher Level Subdivisions

Definition

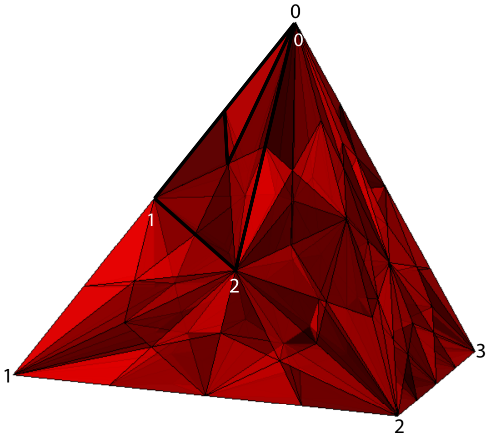
The vertices of a cell A that results from a Barycentric subdivision are labeled by looking at each vertex of K^0 that has is named character a_i then finding the vertex of A that is closest to it using the Euclidean metrix. This vertex of A is then named a_i.



Naming Higher Level Subdivisions

- As we wish to reference cells in further Barycentric subdivisions, K^n, the words become $n \times 4$ matrices where each ith row, $(a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4})$, gives the level-$i$ cell that contains A.
- An example of this: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

An Example of Naming Higher Levels: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$



The Number of Neighbors

A cell can have at most 4 neighbors because it has 4 faces. All cells have 4 neighbors, but some have 3 neighbors if they are boundary cells which are defined as:

Definition

A cell is an *boundary cell* if one of its faces intersects the face of K^0

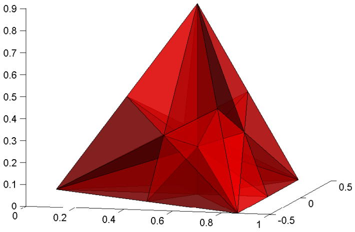
Finding the First Three Neighbors

Corollary

Every cell in $\mathcal{B}^1$ *is a boundary cell.*

Proposition

Any cell $A \in K^n$, $n \geq 1$, *has three neighbors in it's* $(n-1)$ *parent cell.*



Finding the First Three Neighbors

Lemma

Any cell $A \in K^n$ has 3 inner neighbors in the same $(n-1)$-cell obtained by applying the transposition $\sigma_i = (a_{ni}, a_{n(i+1)})$ to the nth row of A for $i = 1, 2, 3$.

Example: Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

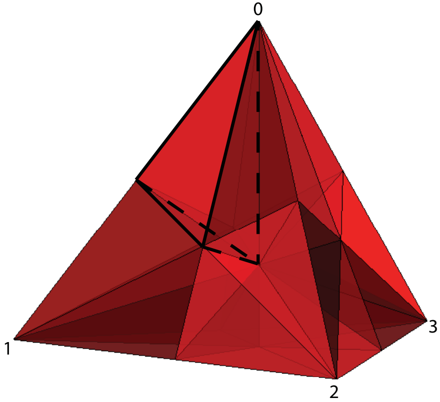
Its 3 neighbors are:

$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{pmatrix}$

This transposition gives us the permutahedron.

Finding the First Three Neighbors - Some Intuition

Let us consider $(0, 1, 2, 3)$ which has neighbors $(1, 0, 2, 3)$, $(0, 2, 1, 3)$, $(0, 1, 3, 2)$

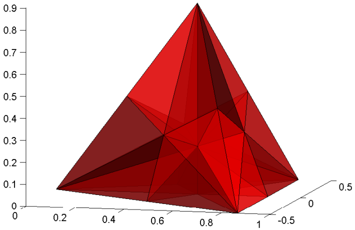


Finding the Fourth Neighbor

Proposition

If $A \notin \partial K^n$ then A has a fourth neighbor in a j cell for some $j \leq n-1$.

Suppose that A, $B \in K^n$ and $A \sim B$ (B is the fourth neighbor), then A's parent is touching B's parent.



Finding the Fourth Neighbor

Using these ideas, we can develop an algorithm to find the fourth neighbor (details omitted)

(1) Find the parent cell that neighbors A.
(2) Find a homomorphism that converts the characters from A's coordinates to the neighbor's coordinates
(3) Apply this homomorphism and obtain the neighbor

The Heat Equation

The heat equation is

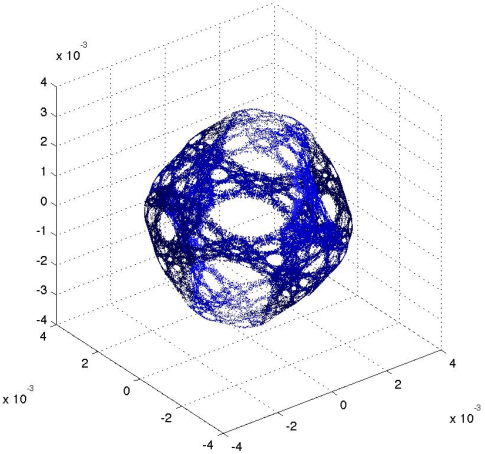
$$\frac{\partial u}{\partial t} - c\Delta u(x,t) = 0 \tag{3}$$

where $t > 0$ where Δ is a Laplacian in the space variable x and c is a positive constant.

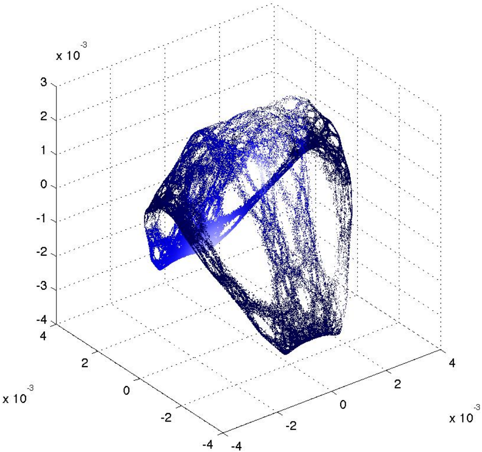
For a level 3 subdivision of a tetrahedron we get the resistance is $\rho = 13.4274$

Eigenfunction Video - Level 4

Eigenfunction Pictures - Level 4



Eigenfunction Pictures - Level 4



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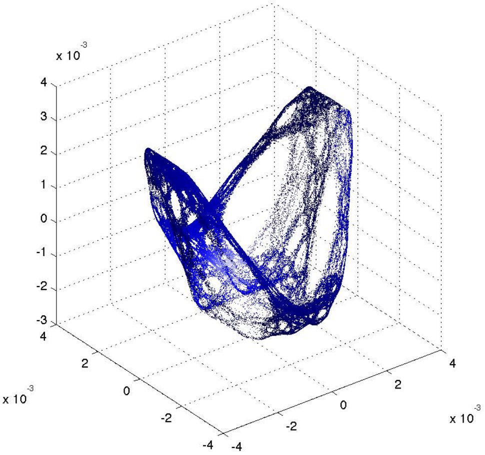

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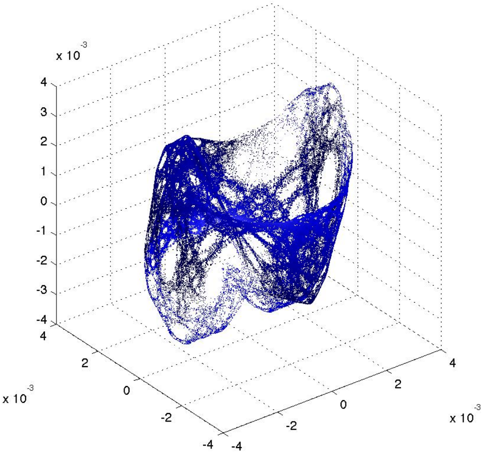

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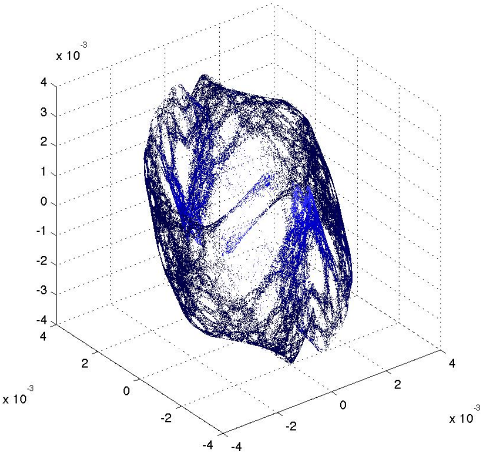

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