Spectrum of the Magnetic Laplacian

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Sierpinski Gasket
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Graph Approximations

And so on...

$V_0$, $V_1$, $V_2$
The Laplacian, $\Delta$, holds information about the nature of a graph.

$$\Delta_n f(x) = \sum_{y \sim x} (f(x) - f(y))$$
Fractal Laplacian

\[ \Delta = \lim_{m \to \infty} 5^m \Delta_m \]
We say $h(x)$ is a harmonic function on $SG$ if $\Delta h(x) = 0$, $\forall x \in SG$.
Harmonic function $h$ with fixed boundary conditions.
Boundary Conditions
A 1–form is the analogue of a vector field, and has values on [directed] edges. The form shown is the exterior derivative of the harmonic function on the previous slide. Its value on the edge from $x$ to $y$ is $h(x) - h(y)$. 
An example of a 1--form that is not the exterior derivative of a harmonic function.
Harmonic Form
\(M_n^\alpha f(x) = \sum_{y \sim x} f(x) - e^{i\alpha A} f(y)\)

\[A(x, y) = h(x) - h(y)\]
Limit of Magnetic Laplacians

\[ M^{\alpha A} = \lim_{m \to \infty} 5^m M_{m}^{\alpha A} \]
Spectrum of the Magnetic Operator

Research goals:

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Spectrum of the Magnetic Laplacian
Research goals:

- Eigenfunctions
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- Eigenfunctions
- Eigenvalues
We are able to find eigenfunctions of $M_{m}^{\alpha A}$. These solutions are found on the cut gasket, however. To bring these solutions back to $SG$ we need to do a gluing similar to a gluing in a Calculus 1 course.
Say we want to join two functions $f_1$ and $f_2$ at point $p$ where $p = l_1 \cap l_2$. 
Gluing from Calc 1

\[ f : l_1 \cup l_2 \rightarrow \mathbb{R} \]

- Continuous: \( f_1(p) = f_2(p) \)
- Differentiable: \( f_1(p) = f_2(p) \) and \( f'_1(p) = f'_2(p) \)
- Twice Differentiable: \( f_1(p) = f_2(p) \), \( f'_1(p) = f'_2(p) \), and \( f''_1(p) = f''_2(p) \)
- etc.
Suppose we have functions on subcells of $SG$ that are to be joined at point $p = S_1 \cap S_2$.

\[
f_1 : S_1 \to \mathbb{R} \quad f_2 : S_2 \to \mathbb{R}
\]

\[
f : S_1 \cup S_2 \to \mathbb{R}
\]
Gluing Functions on Subcells of $SG$

Our joined function $f$ will have a continuous Laplacian if

1. $f_1(p) = f_2(p)$
2. $\partial_n f_1(p) + \partial_n f_2(p) = 0$
3. $\Delta f_1(p) = \Delta f_2(p)$

Note: If $f_1$ and $f_2$ are eigenfunctions of $\Delta$ with the same eigenvalue then $1 \implies 3$. We then only need to check conditions 1 and 2.
Normal Derivative

\[ \partial_n f(x) := \lim_{m \to \infty} \left( \frac{5}{3} \right)^m [2f(x) - f(y_i) - f(z_i)] \]
Suppose we are on a subcell of $SG$ and the magnetic field is $dA$ for a harmonic function $A$. Then our operator becomes

$$M_m^{\alpha A} = e^{i\alpha A} \Delta e^{-i\alpha A}.$$  

It follows that if $f$ is an eigenfunction of $\Delta$ on this subcell with eigenvalue $\lambda$ then $g = e^{-iA}f$ is an eigenfunction of $M_m^{\alpha A}$ with the same eigenvalue $\lambda$.

We obtain eigenfunctions of $M_m^{\alpha A}$ on all of $SG$ by gluing the eigenfunctions of subcells.
Infinite Ladder Sierpinski Fractal Fold
Infinite Ladder Sierpinski Fractal