Determining the Spectrum of the Laplacian on $3N$-Gaskets

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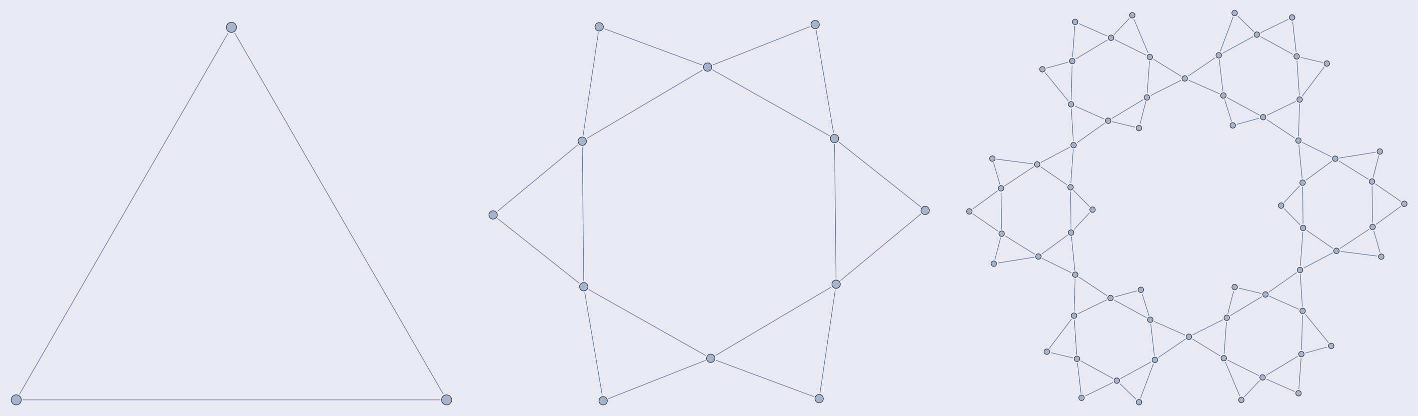
Introduction

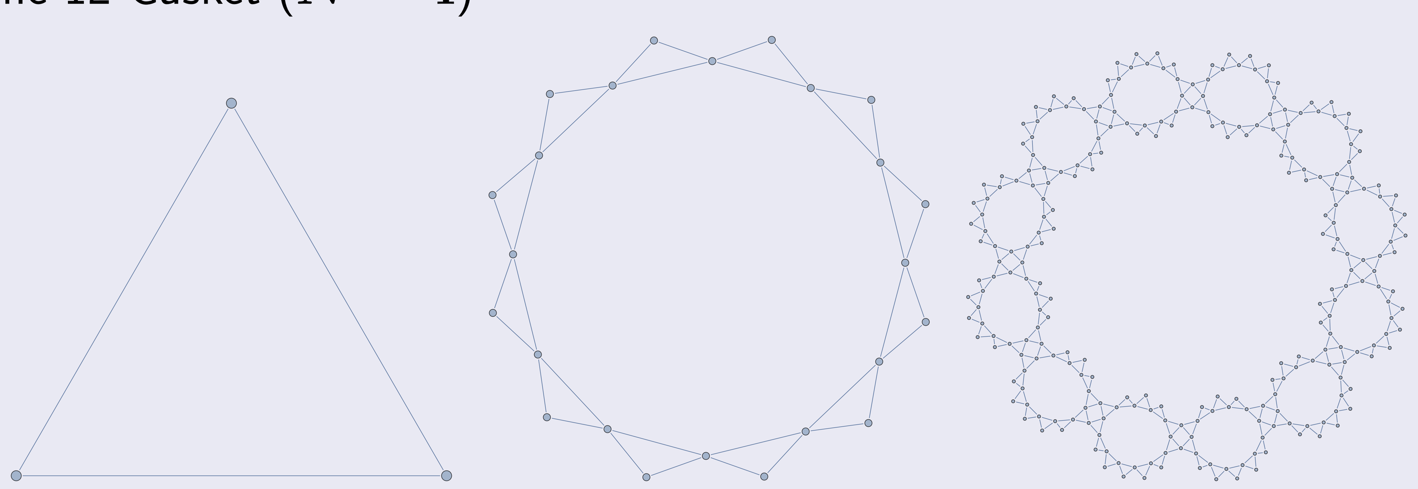
The Laplacian is the central object of analysis on fractals. While most work on the Laplacian has been focused on computing the Laplacian's spectrum on specific fractals, we instead find the spectrum on an entire class of fractals—the $3N$-Gaskets. This is the class of finitely ramified fractal $3N$-Gons that are the attractors of iterated function systems containing $3N$ contraction mappings. For example, the 3-Gasket is the Sierpinski triangle. The Laplacian on the fractal, and thus its eigenvalues, must be studied by examining the graph Laplacian on approximating graphs. We find a function $R(z)$ relating the Laplacian eigenvalues on consecutive graph approximations. We use $R(z)$ along with the eigenvalues of the Laplacian on graph approximations to derive part of the spectrum on the next level. The rest of the spectrum is found using an "exceptional set"— a set of values which we provide for arbitrary N. By repeating this infinitely many times, the spectrum of the Laplacian on the fractal can be found.

Construction of the $3N$-Gaskets

- The $3N$-Gaskets are the result of a repeated application of an iterated function system of $3N$ contraction mappings to a triangle.
- The Hexagasket ($N = 2$)

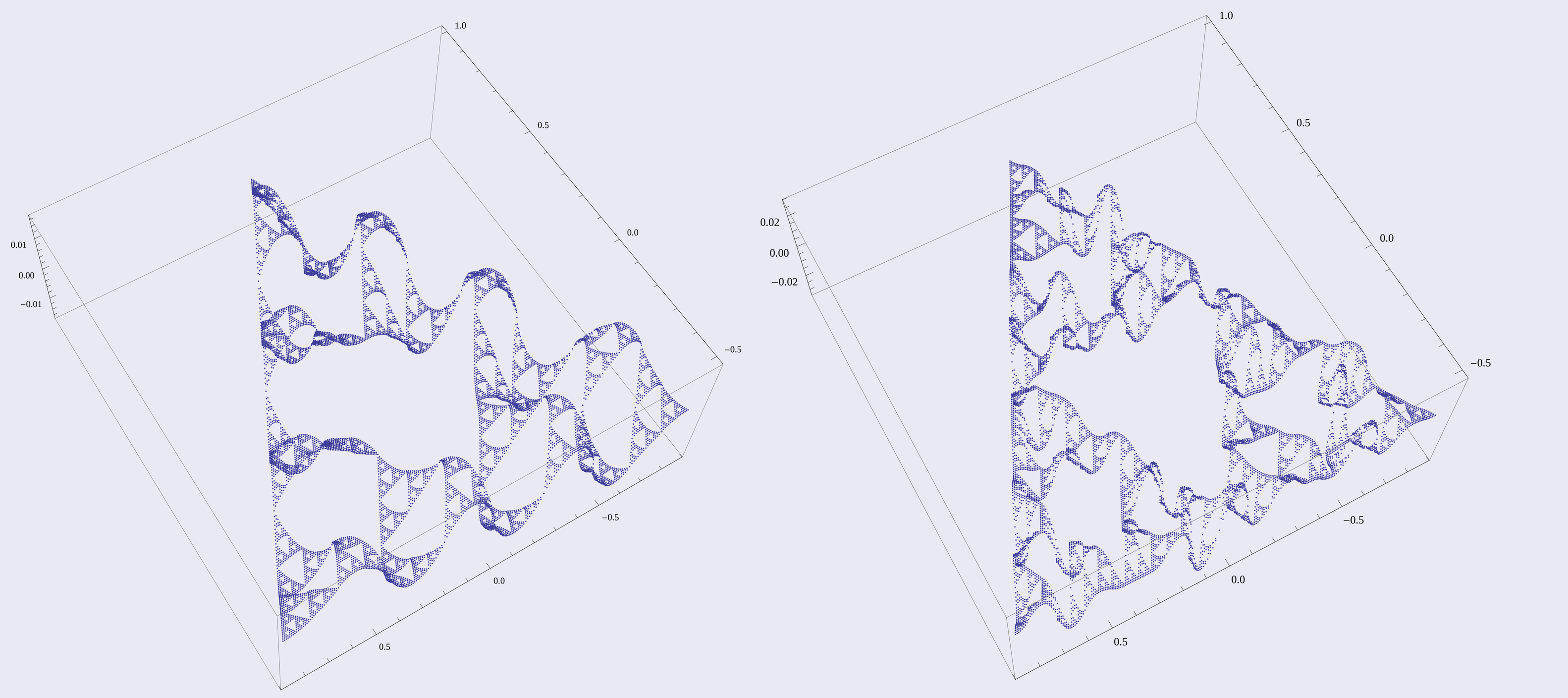


- The 12-Gasket ($N = 4$)



Eigenplot for $N = 1$

Through graphing, we can easily see that eigenfunctions may be interpreted as describing vibration modes on fractals. For the Sierpinski Triangle,



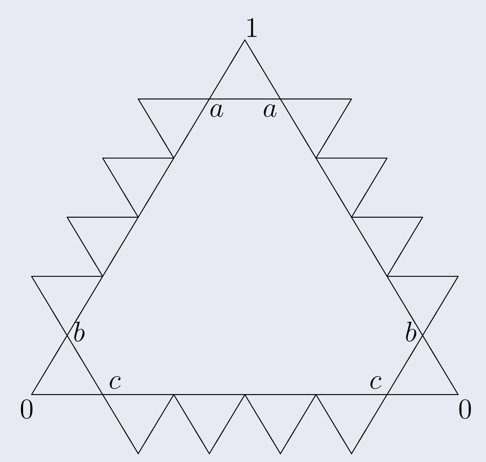
Spectral Decimation

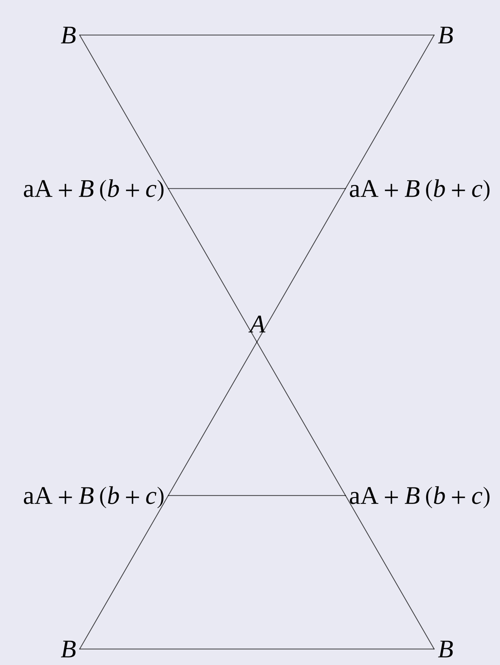
- We would like to find a function $R(z)$ relating the eigenvalues on consecutive levels of the fractal and apply $R^{-1}(z)$ repeatedly to the eigenvalues on the zeroth level of the fractal, $\sigma(M_0)$, to get the eigenvalues on any level.
- Unfortunately, this does not work and we must gather additional tools and use a proposition from a previous UConn REU to find the spectrum
- The functions and sets of eigenvalues are required:
  1. $R(z)$—"The Spectral Decimation Function"
  2. $\sigma(M_0)$ and $\sigma(M_1)$—sets of eigenvalues corresponding to Δ_0 and Δ_1 respectively
  3. $\phi(z)$ and $\sigma(D)$—a rational function and a set of eigenvalues of Δ_n corresponding to Dirichlet boundary conditions used to find an exceptional set of values, $E(M_0, M)$

Calculation of $R(z)$

- We want to relate the eigenvalues on arbitrary levels n and $n + 1$ of the graph
- Thus, we consider any junction point, give it the value A, and evaluate the Graph Laplacian on the two levels
- But we need the values at the vertices adjacent to the junction point
- Accordingly, we consider a simpler problem and find eigenfunctions that live on the "sawtooth graph" with boundary values $1, 0, 0$
- Taking linear combinations of these eigenfunctions, we find the values of the eigenfunction at the vertices
- This gives us the following form for the spectral decimation function:

$$R(z) = \frac{a + b + c + z - 1}{b + c}$$





$R(z)$ Final Form

- Thus, we have the following form for $R(z)$:
- If N is even then

$$R(z) = \frac{(z-1)\sqrt{z}U_{N-1}\left(\sqrt{z}\right)(2T_N(1-2z)+2U_{N-1}(1-2z)+1)}{T_N\left(\sqrt{z}\right)}$$

- If N is odd then

$$R(z) = \frac{\sqrt{z}T_N\left(\sqrt{z}\right)(2T_N(1-2z)+2U_{N-1}(1-2z)+1)}{U_{N-1}\left(\sqrt{z}\right)}$$

The Spectrum of the Laplacian on the Fractal

- For N even,

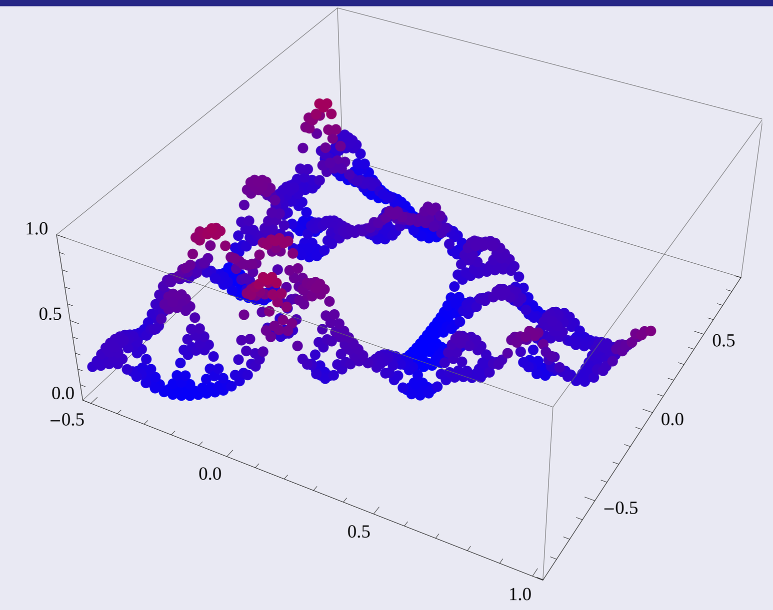
$$\sigma(\Delta) = \left\{\frac{3}{2}\right\} \cup \Bigg(\bigcup_{k=0}^{\infty} R_{-k}\Bigg(\left\{\sin^2\left(\frac{m\pi}{3N}\right) : m = 0, ..., 3N - 1 \text{ and } 3 \nmid m\right\} \cup \left\{z : T_N(\sqrt{z}) - 2(z-1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0\right\} \cup \left\{\cos^2\left(\frac{m\pi}{N}\right) : m = 1, ..., \frac{N}{2}\right\}\Bigg)\Bigg)$$

- For N odd,

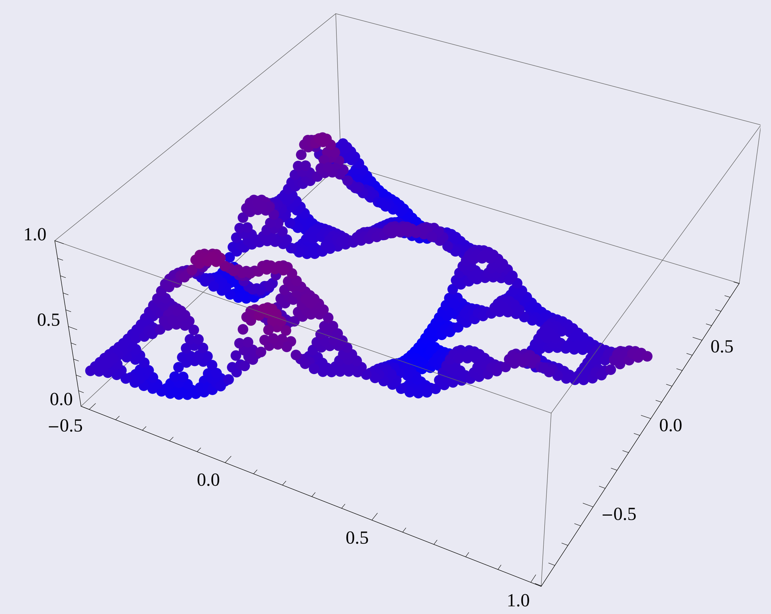
$$\sigma(\Delta) = \left\{\frac{3}{2}\right\} \cup \Bigg(\bigcup_{k=0}^{\infty} R_{-k}\Bigg(\left\{\sin^2\left(\frac{m\pi}{3N}\right) : m = 0, ..., 3N - 1 \text{ and } 3 \nmid m\right\} \cup \left\{z : 2T_N(1-2z) + 2U_{N-1}(1-2z) + 1 = 0\right\}) \cup \left\{\cos^2\left(\frac{\left(m-\frac{1}{2}\right)\pi}{N}\right) : m = 1, ..., \frac{N-1}{2}\right\}\Bigg)\Bigg)$$

Sierpinski Gasket Heat Dispersion Plots

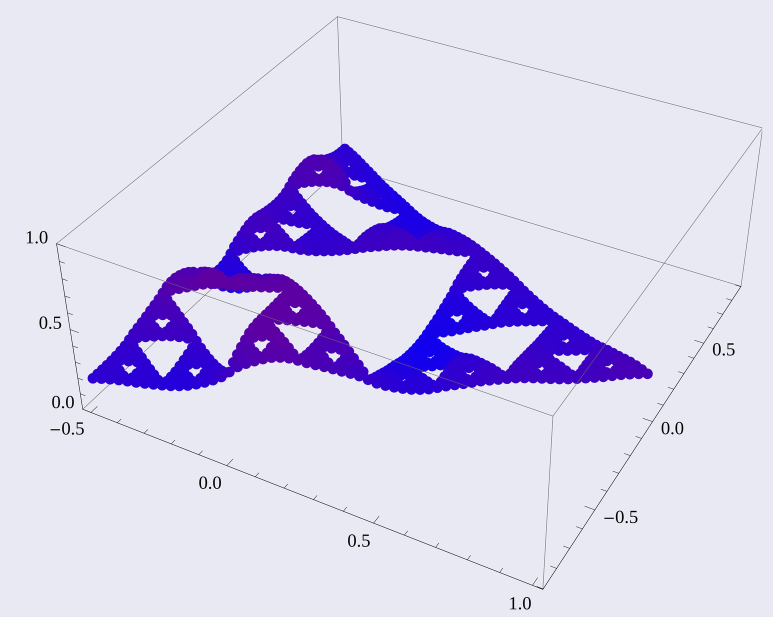
- Using the spectrum, $\sigma(\Delta)$, we may find the heat kernel, and thus, find a solution to the heat equation on the $3N$-Gasket
- Accordingly, may simulate the flow of heat across a $3N$-Gon when heat is supplied to the boundary points
- Heat dispersion for increasing t:



$t = 6$



$t = 10$



$t = 20$

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