Geodesics and a Riemannian Metric on Harmonic Sierpinski Gaskets

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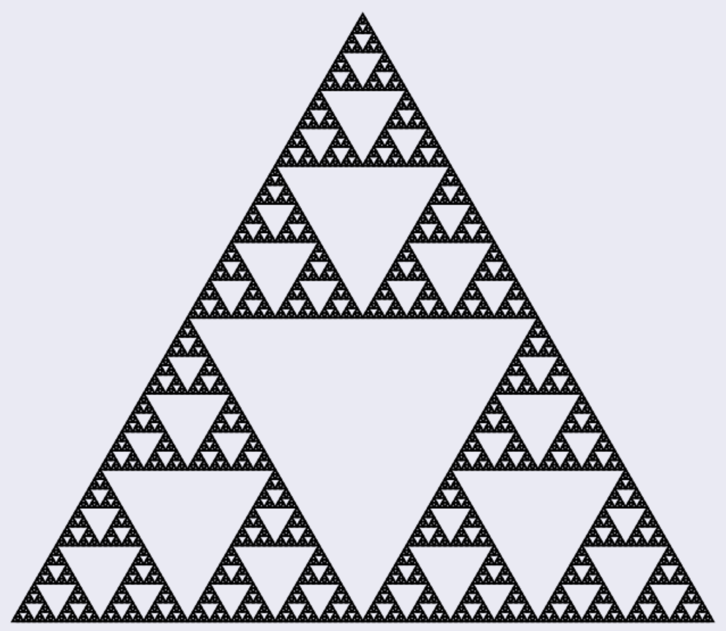
Introduction

On the classical Sierpinski gasket, there is a way to construct a Laplacian differential operator and use it to do calculus. However, when the gasket is embedded in $\mathbb{R}^2$ in the usual way, the restriction of the calculus structure of $\mathbb{R}^2$ to the set does not give the same kind of structure.

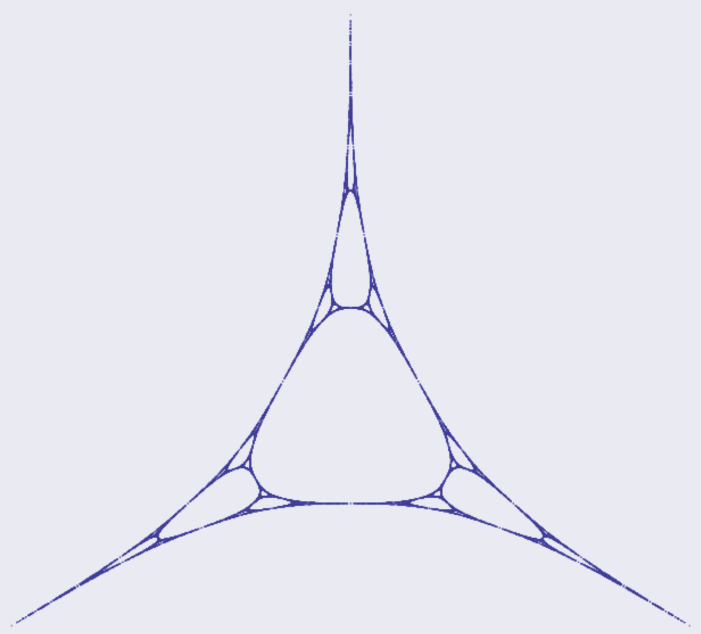
Kigami has shown that there is a way to embed the gasket in $\mathbb{R}^2$ such that treating the image like a Riemannian submanifold recovers the correct structure for doing analysis on the gasket (1993, 2008). We extend his results to the Sierpinski gasket on N vertices and an embedding in $\mathbb{R}^{N-1}$. The main task is to prove that there are geodesics between pairs of boundary vertices, the lengths of which are given by a Riemannian-type metric.

Construction of the Sierpinski Gasket SG_N

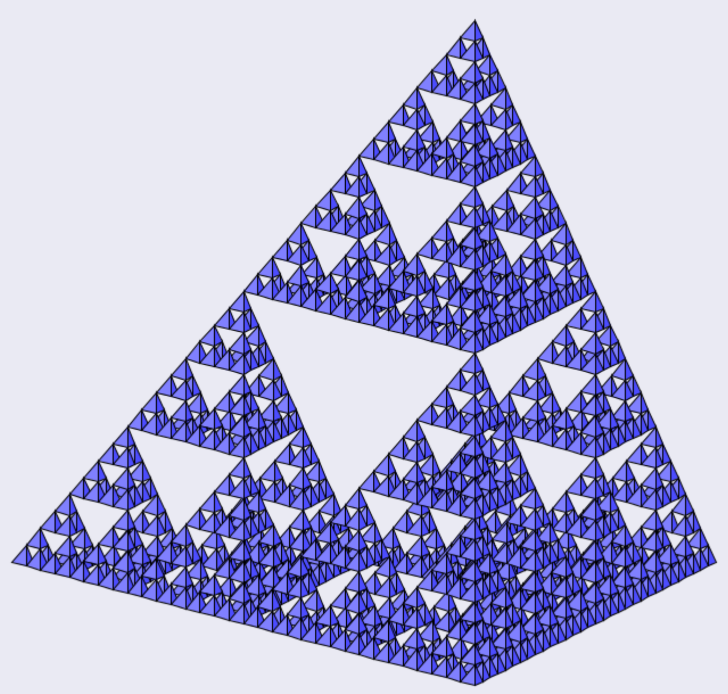
- SG_N is the attractor of an iterated function system, i.e. there is a set of functions H_i such that
$$\bigcup_{i=1}^{N} H_i(SG_N) = SG_N.$$
- Typically, $H_i(x) = \frac{1}{2}(x - p_i) + p_i$, where p_i are the vertices of an $(N-1)$-dimensional simplex (see left below).
- For the harmonic gasket, K_N (see below right), $H_i(x) = T_i(x - p_i) + p_i$. If the origin is at the barycenter of the simplex, T_i is the linear map that contracts the simplex by a factor of $\frac{N}{N+2}$ in the direction of p_i and $\frac{1}{N+2}$ in all orthogonal directions.



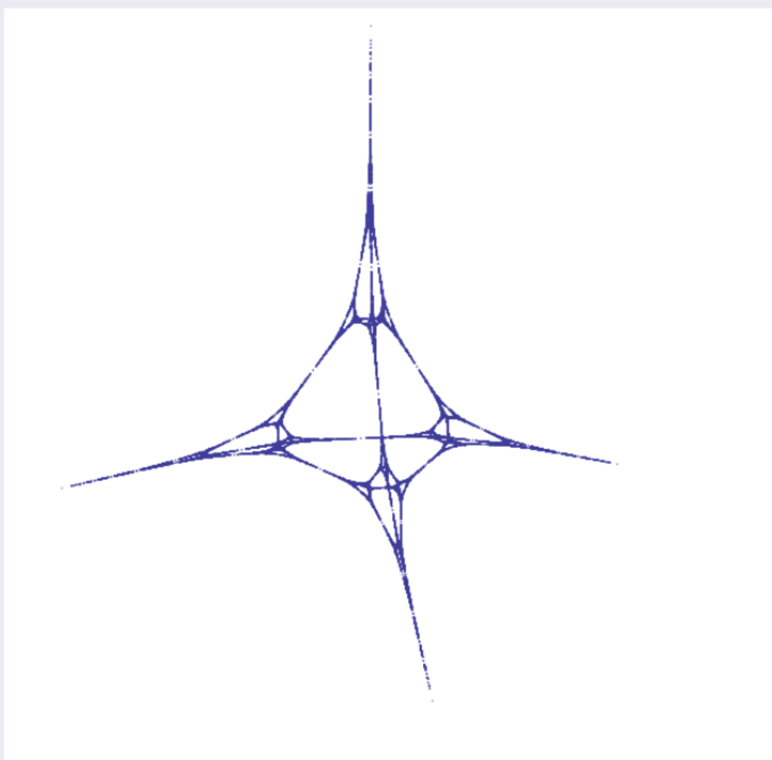
SG_3 in regular coordinates



Harmonic embedding of SG_3



SG_4 in regular coordinates



Harmonic embedding of SG_4

Connecting the Boundary of K_N with Geodesic Curves

Let C be the attractor of the iterated function system $\{H_1, H_2\}$, so $C = H_1(C) \cup H_2(C)$.

Theorem C is a convex curve of finite length lying in a plane P.

Theorem All points in the projection of the harmonic Sierpinski gasket onto the plane P lie on the concave side of C.

Corollary The shortest path in the harmonic gasket between the fixed points of H_1 and H_2 is C.

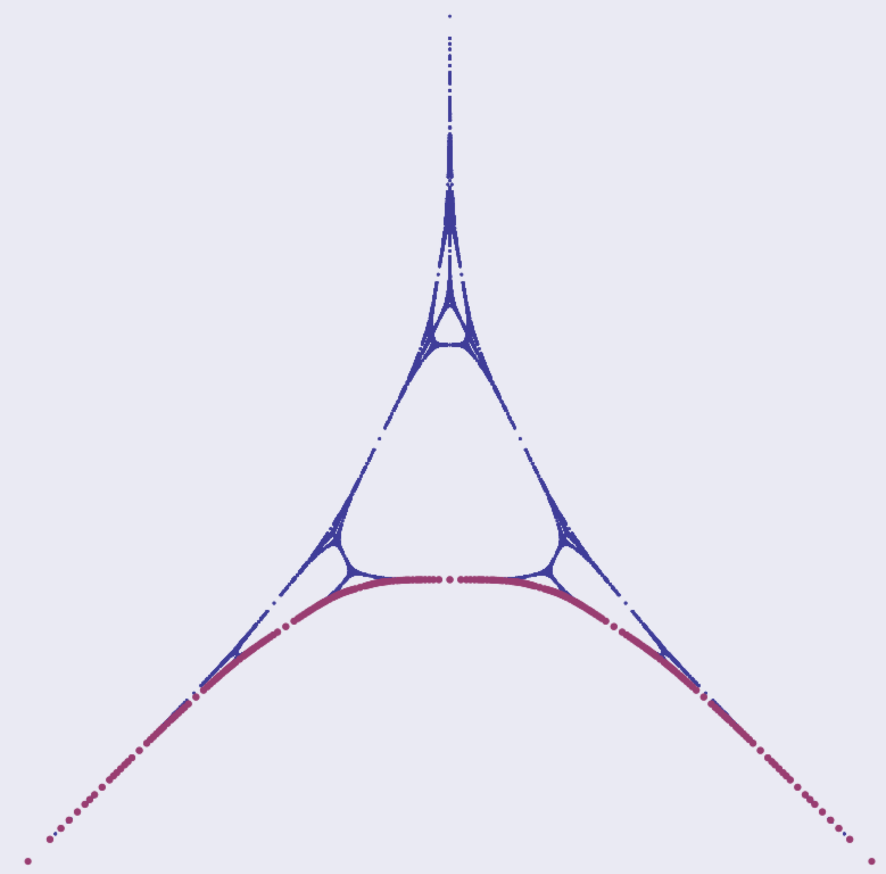


Figure : Projection of SG_4 (blue) onto the plane P, and the curve C (red)

Properties of C and Construction of de Rham Curves

- We prove C is planar by showing that the plane P is a closed set and invariant under the two maps H_1 and H_2, and therefore contains the attractor of the IFS $\{H_1, H_2\}$.
- Many other properties of C follow from
  Theorem C is a de Rham curve with $\omega = \frac{N}{2(N+2)}$.
- De Rham curves are constucted by subdividing the sides of a regular n-gon, into three pieces with ratio $\omega : 1 - 2\omega : \omega$ and constructing a new convex polygon with $2n$ sides, whose vertices are those formed by the subdivision. For example, Figure 2 shows a de Rham curve on a polygon with vertices a, b and c:

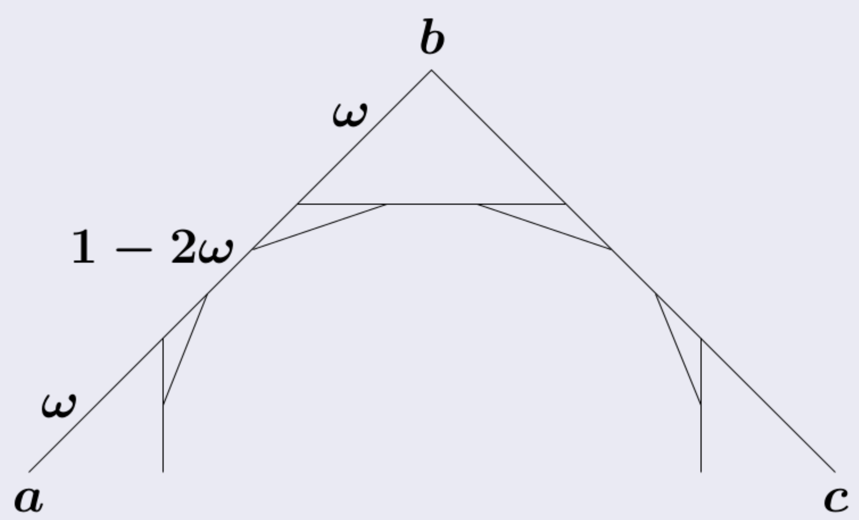


Figure : Example of a de Rham Curve up to two iterations

Theorem [Protasov '04] A de Rham curve is C^1 if $\omega \le 1/3$, and is differentiable at all but countably many points if $1/3 < \omega < 1/2$.

Corollary C is C^1 if $N = 2, 3$ or 4, and its derivative has countably many points of discontinuity if $N \ge 5$.

A Geodesic Distance on the Harmonic Sierpinski Gasket

- Let x and y be vertices of a cell in K_N. Since C is the shortest curve between p_1 and p_2, the shortest path between x and y is the image of C along the corresponding edge of this cell.
- If x is a vertex of one cell and y is a vertex of another cell, then the shortest curve between them in K_N is a finite union of images of C which make up the edges of cells on the path from x to y. (See Figure 3.)
- By a limiting procedure, the shortest curve in K_N between arbitrary points x and y is a countable union of images of C. We let $d_*(x, y)$ denote the length of this curve.

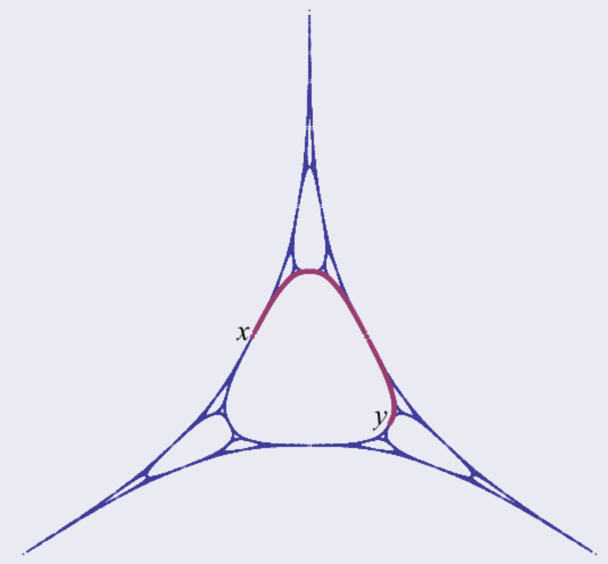


Figure : Shortest path from x to y in K (red)

Riemanian Metric Measure Structure and Heat Kernel

If $x \in C$ is not a vertex, then there is a unique sequence $w_1, w_2, \dots$ and $x = \lim_{m\to\infty} H_{w_1} \circ H_{w_2} \circ \cdots \circ H_{w_m}(K_N)$. Let $T_m(x) = T_{w_1} T_{w_2} \cdots T_{w_m}$ and define
$$Z(x) = \lim_{m\to\infty} \frac{T_m(x) T_m^t(x)}{\|T_m(x)\|_{HS}^2},$$
provided that $Z(x)$ exists, where $\|T_m\|_{HS}$ is the Hilbert-Schmidt norm (i.e $\|T_m\|_{HS}^2$ is the sum of the eigenvalues of $T_m T_m^t$, counted with multiplicity).

Conjecture For all such x, $Z(x)$ exists and is the projection onto the tangent direction to C at x.

This would prove that Z is like a Riemannian metric which gives the geodesic distance d_*, in that if $\gamma\colon [0, 1] \to C$ is Lipschitz, then
$$\ell(C) = \int_0^1 | < \gamma'(t), Z(\gamma(t))\gamma'(t) > |^{1/2} dt,$$
where $\ell(C)$ is the length of the curve C. Combined with results of Kigami [Math. Ann.'08], this would imply that the heat kernel of the Kusuoka Laplacian on K_N satisfies Gaussian estimates, i.e. there are constants c_1, c_2, c_3 and c_4 such that
$$\frac{c_1}{v(B_{\sqrt{t}}(x, d_*))} exp\left(-c_2 \frac{d_*(x,y)^2}{t}\right) \le p_v(t, x, y) \le \frac{c_3}{v(B_{\sqrt{t}}(x, d_*))}(x, d_*)) exp\left(-c_4 \frac{d_*(x,y)^2}{t}\right)$$
for any $t \in (0, 1]$ and any $x \in K_N$.

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