Geodesics and a Riemannian Metric on Harmonic Sierpinski Gaskets

Sara Chari, Joshua Frisch

Introduction

On the classical Sierpinski gasket, there is a way to construct a Laplacian differential operator and use it to do calculus. However, when the gasket is embedded in $\mathbb{R}^2$ in the usual way, the restriction of the calculus structure of $\mathbb{R}^2$ to the set does not give the same kind of structure.

Kigami has shown that there is a way to embed the gasket in $\mathbb{R}^2$ such that treating the image like a Riemannian submanifold recovers the correct structure for doing analysis on the gasket (1993, 2008). We extend his results to the Sierpinski gasket on $N$ vertices and an embedding in $\mathbb{R}^{N-1}$. The main task is to prove that there are geodesics between pairs of boundary vertices, the lengths of which are given by a Riemannian-type metric.

Construction of the Sierpinski Gasket $SG_N$

$SG_N$ is the attractor of an iterated function system, i.e. there is a set of functions $H_i$ such that

$$\bigcup_{i=1}^N H_i(SG_N) = SG_N.$$  

Typically, $H_i(x) = \frac{1}{3}(x - p_i) + p_i$, where $p_i$ are the vertices of an $(N-1)$-dimensional simplex (see below).

For the harmonic gasket, $SG_N$ (see below right),

$$H_i(x) = T_i(x - p_i) + p_i$$  

when the origin is at the barycenter of the simplex, $T_i$ is the linear map that contracts the simplex by a factor of $\frac{1}{N+2}$ in the direction of $p_i$ and $\frac{1}{N+2}$ in all orthogonal directions.

Properties of $C$ and Construction of de Rham Curves

- We prove $C$ is planar by showing that the plane $P$ is a closed set and invariant under the maps $H_1$ and $H_2$, and therefore contains the attractor of the IFS $\{H_1, H_2\}$.
- Many other properties of $C$ follow from Theorem $C$ is a de Rham curve with $\omega = \frac{N}{2(N+2)}$.
- De Rham curves are constructed by subdividing the sides of a regular $n$-gon, into three pieces with ratio $\omega = 1 - \omega$ : $\omega$ and constructing a new convex polygon with $2n$ sides, whose vertices are those formed by the subdivision. For example, Figure 2 shows a de Rham curve on a polygon with vertices $a$, $b$ and $c$.

Connecting the Boundary of $K_N$ with Geodesic Curves

Let $C$ be the attractor of the iterated function system $\{H_1, H_2\}$, so $C = H_1(C) \cup H_2(C)$.

**Theorem** $C$ is a convex curve of finite length lying in a plane $P$.

**Theorem** All points in the projection of the harmonic Sierpinski gasket onto the plane $P$ lie on the concave side of $C$.

**Corollary** The shortest path in the harmonic gasket between the fixed points of $H_1$ and $H_2$ is $C$.

A Geodesic Distance on the Harmonic Sierpinski Gasket

- Let $x$ and $y$ be vertices of a cell in $K_N$. Since $C$ is the shortest curve between $x_1$ and $y_1$, the shortest path between $x$ and $y$ is the image of $C$ along the corresponding edge of this cell.
- If $x$ is a vertex of one cell and $y$ is a vertex of another cell, then the shortest curve between them in $K_N$ is a finite union of images of $C$ which make up the edges of cells on the path from $x$ to $y$ (See Figure 3.)
- By a limiting procedure, the shortest curve in $K_N$ between arbitrary points $x$ and $y$ is a countable union of images of $C$. We let $d_{s}(x, y)$ denote the length of this curve.

Figure: Shortest path from $x$ to $y$ in $K$ (red)

Riemannian Metric Measure Structure and Heat Kernel

If $x \in C$ is not a vertex, then there is a unique sequence $w_1, w_2, \ldots$ and $x = \lim_{m \to \infty} H_{w_1} \circ H_{w_2} \circ \cdots \circ H_{w_m}(K_N)$. Let $T_m(x) = T_{w_1} T_{w_2} \cdots T_{w_m}$ and define

$$Z(x) = \lim_{m \to \infty} \frac{\|T_m(x)\|_{FS}}{\|T_m(x)\|_{HS}}$$

provided that $Z(x)$ exists, where $\|T_m\|_{FS}$ is the Hilbert-Schmidt norm (i.e. $\sum \lambda_i$ is the sum of the eigenvalues of $T_m^* T_m$ counted with multiplicity).

**Conjecture** For all such $x$, $Z(x)$ exists and is the projection onto the tangent direction to $C$ at $x$.

This would prove that $Z$ is like a Riemannian metric which gives the geodesic distance $d_s$, in that if $\gamma : [0, 1] \to C$ is Lipschitz, then

$$\ell(C) = \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} \, dt$$

where $\ell(C)$ is the length of the curve $C$. Combined with results of Kigami [Math. Ann. '08], this would imply that the heat kernel of the Kusuoka Laplacian on $K_N$ satisfies Gaussian estimates, i.e. there are constants $c_1, c_2, c_3$ and $c_4$ such that

$$\frac{c_1}{c_2} e^{-c_3 \frac{d_s(x,y)^2}{t}} \leq \frac{c_4}{c_2} e^{-c_3 \frac{d(x,y)^2}{t}} \leq \frac{c_4}{c_2} e^{-c_3 \frac{d(x,y)^2}{t}}$$

for any $t \in (0, 1]$ and any $x \in K_N$.

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