Magnetic Spectral Decimation on the Diamond Fractal

An Exploration of Eigenvalues of the Magnetic Laplacian

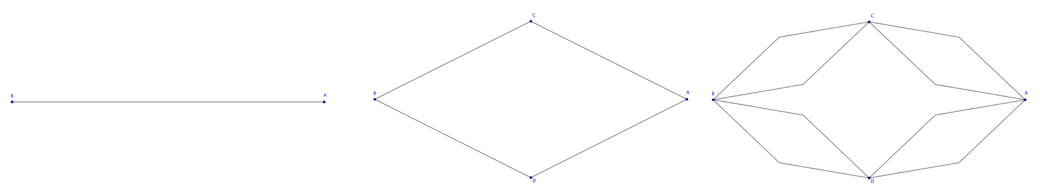
Aubrey Coffey, Madeline Hansalik, Stephen Loew

Department of Mathematics, University of Connecticut

Abstract

The Laplacians for a large class of self-similar fractals and fractal graphs exhibit a property called spectral decimation, in which the spectra of different levels of approximation are related by a dynamical system involving a rational function. Expanding upon the work of Malozemov and Teplyaev [1], we extend some aspects of the spectral decimation method from the Laplacian operator to a magnetic Laplacian operator, and use this to numerically investigate properties of the magnetic spectrum of this operator on the diamond fractal. In particular, we identify the correct unitary transformations and projections to obtain the aforementioned rational functions.

Diamond Fractal



- The $(n+1)^{\text{th}}$ approximating graph is generated by replacing each edge in the graph of the n^{th} level with a copy of the level 1 graph (i.e. lines become diamonds)
- We can represent this operation using an iterated function system (IFS) of four contraction maps f_i, $1 \le i \le 4$, where each f_i maps the diamond to a smaller diamond. The diamond fractal is the unique, non-empty, compact set invariant under the IFS: $K = \bigcup f_i(K)$

Laplacian

- Laplacian
  - The graph Laplacian at level n is an operator on functions given by

$$\Delta_n u(x) = \sum_{y \sim x} \big(u(x) - u(y)\big)$$

  - This can be represented in matrix form as the difference of the graph's degree matrix and adjacency matrix. At level zero we have:

Degree Matrix	Adjacency Matrix	Laplacian Matrix
$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

  - For our work, we normalize by replacing the a_{ij} entry with $\frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}$. This changes the eigenvectors but not the eigenvalues of the matrix.
- Laplacian on the Diamond Fractal
  - At any level n the graph Laplacian is a block matrix:

$$\begin{bmatrix} A & B \\ B^t & D \end{bmatrix}$$

    - The A block corresponds to vertices from the $(n-1)^{\text{th}}$ level.
    - The D block corresponds to new vertices introduced at the n^{th} level.
    - The A and D blocks are diagonal.
    - B and B^t correspond to connections between levels $n-1$ and n.
  - It is known that $4^n\Delta_n$ converges to an operator that is the correct replacement for the usual Euclidean Laplacian on the Diamond Fractal.

Spectral Decimation for the Laplacian

On the Diamond fractal it is known that we can relate the spectrum of Δ_n to Δ_{n-1} via the Schur complement. Making the standard computation

$$(\Delta_n - \lambda)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A-\lambda & B \\ B^t & D-\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we see that if λ is not an eigenvalue of D then

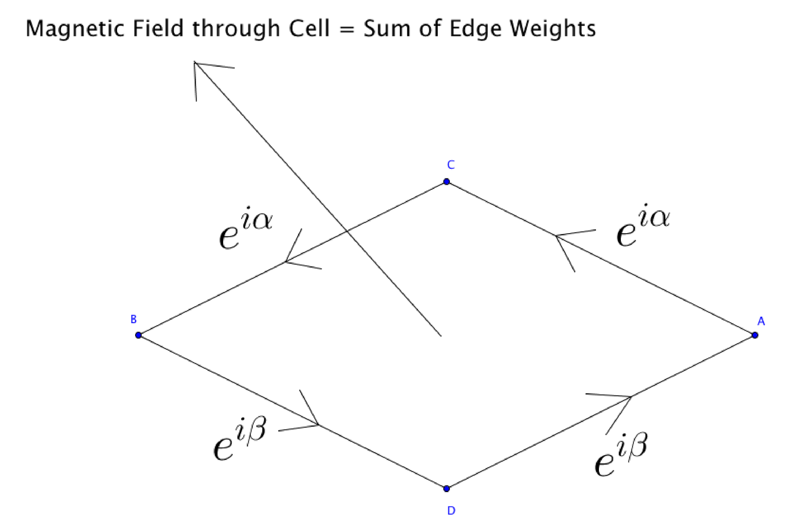
$$S_\lambda = A - \lambda - B^t(D-\lambda)B = 0$$

where S_λ is the Schur complement.

Theorem 1. *For the Diamond Fractal there exist rational functions $\varphi_0(\lambda)$ and $\varphi_1(\lambda)$ such that $S_\lambda = \varphi_0(\lambda)\Delta_{n-1} - \varphi_1(\lambda)I$.*

Corollary 1. *If λ is not an eigenvalue of D and $\varphi_0(\lambda) \neq 0$ then λ is an eigenvalue of Δ_n if and only if $\frac{\varphi_1(\lambda)}{\varphi_0(\lambda)}$ is an eigenvalue of Δ_{n-1}.*

Magnetic Laplacian



- Magnetic Laplacian
  - Approximating graph becomes a directed graph.
  - Edges are weighted by $e^{i\theta}$ in the direction of the edge and $e^{-i\theta}$ in the opposite direction, for some $\theta \in \mathbb{R}$.
  - By convention, the field strength through a hole is the counterclockwise oriented sum of the θ values on the edges around the hole.
  - The Magnetic Laplacian is then given by

$$M_n u(x) = \sum_{y \sim x} \big(u(x) - e^{i\theta_{xy}}u(y)\big),$$

    where θ_{xy} corresponds to the edge from x to y.
  - We may still make a block decomposition as we did for the Laplacian; blocks A and D are unchanged, the $e^{i\theta}$ factors appear in the off-diagonal blocks, which are now B and B^*.
- Physically, the spectrum of the Magnetic Laplacian corresponds to energy levels of a quantum particle confined to the Diamond Fractal and under the influence of a magnetic field.

Spectral Decimation and Magnetic Laplacian

A variant of Spectral Decimation still works for M_n on the Diamond Fractal, but it is not as simple as it was in the case of the Laplacian. Instead of fixed functions φ_0 and φ_1 we have functions that depend on the magnetic field strength.

The computation of the Schur complement for the diamond configuration with all edge weights equal to $e^{i\gamma}$ is as follows:

$$M_1 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \text{ with } A = D = (1-\lambda)I \text{ and } B = \begin{bmatrix} \frac{-e^{i\gamma}}{2} & \frac{-e^{-i\gamma}}{2} \\ \frac{-e^{-i\gamma}}{2} & \frac{-e^{i\gamma}}{2} \end{bmatrix}.$$

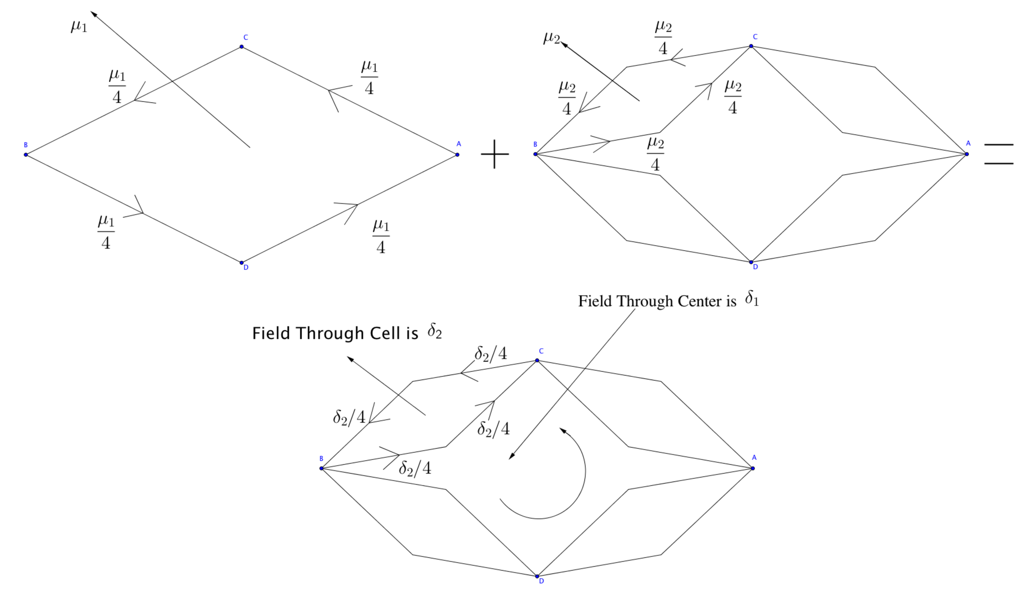
Then

$$\begin{aligned} S_\lambda &= \begin{bmatrix} 1-\lambda-\frac{1}{2(1-\lambda)} & \frac{-\cos(2\gamma)}{2(1-\lambda)} \\ \frac{-\cos(2\gamma)}{2(1-\lambda)} & 1-\lambda-\frac{1}{2(1-\lambda)} \end{bmatrix} \\ &= \frac{\cos(2\gamma)}{2(1-\lambda)}M_0 - \left(\frac{-2\lambda^2+4\lambda-1+\cos(2\gamma)}{2(1-\lambda)}\right)I \end{aligned}$$

It does not immediately follow that this method can be used to reduce M_n to M_{n-1}, but there is a method to do this. In order to prove this, we generalize results of [1], where spectral decimation on operators corresponding to pieces of a graph can be "glued" together to obtain spectral decimation on the operator for the whole graph. The additional feature we need is that each piece may be gauge-transformed, i.e. conjugated by a unitary matrix, before gluing. By solving an auxiliary problem about writing a general field as a sequence of gauge transforms from level to level, we obtain spectral decimation for the Diamond fractal.

Gauge transforms from level to level

In order to use our spectral decimation approach we must write our magnetic field as a sequence of fields in which the level k field acts as a gauge transform on all cells smaller than level k. The following diagram illustrates this for two levels.



Generalizing from this diagram we show

Lemma 1. *Fix a level n. Given a magnetic field in which the field strength through a hole depends only on the level of the hole, let δ_j denote the field strength through each j^{th} level hole, and define*

$$\mu_j = \delta_j + \sum_{i=j+1}^{n} 2^{2i-(2j+1)}\delta_i.$$

Define a field of strength μ_j on each j-level cell, and extend it to act as a gauge transform on all smaller cells. Then the original field is the sum of the μ_j fields.

From our gauge-transformed gluing result, we then have the following spectral decimation theorem on the diamond fractal.

Theorem 2. *Suppose $\{\lambda_n\}$ is a sequence such that for each n, $\lambda_n \neq 1$ and $\mu_n \notin \frac{\pi}{2}\mathbb{Z}$. Then for each n, λ_n is an eigenvalue of M_n if and only if $R(\lambda_n, \mu_n)$ is an eigenvalue of M_{n-1}, where*

$$R(\lambda, \mu) = \frac{4\lambda - 2\lambda^2 - 1}{\cos(\mu/2)} + 1.$$

Remark 1. The exceptional case $\lambda_n = 1$ does correspond to eigenvalues, and we can calculate their multiplicity. Another exceptional case occurs when $\varphi_0(\lambda) = 0$; this does not immediately lend to eigenvalues.

This theorem gives an algorithm for finding eigenvalues of the Laplacian on the Diamond Fractal. For a given magnetic field defined by the sequence $\{\delta_i\}_{i=1}^n$, compute $\{\mu_i\}_{i=1}^n$. Then for each i:

1. For every eigenvalue $\lambda_k^{(i-1)}$ of M_{i-1}, find its two preimages under $R(\cdot, \mu_i)$.
2. Incorporate $\frac{1}{3}(4-4^i)+4^i$ copies of the exceptional eigenvalue $\lambda = 1$.
3. Re-scale all eigenvalues by 4^n.
4. Take the limit as $n \to \infty$.

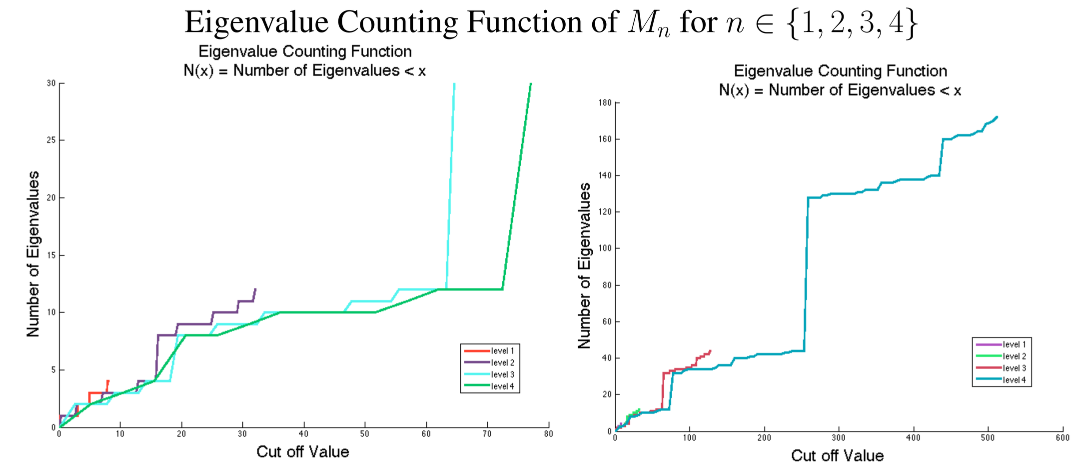
Steps (1) and (2) generate the un-normalized spectrum on level n. Step (3) renormalizes the eigenvalues. Recall $4^n\Delta_n$ converges to the Laplacian on the Diamond Fractal; the same scaling holds for the magnetic Laplacian.

As a special case we consider δ_j to be proportional to area of the j^{th} level cell. For a suitable embedding of the fractal, the area occupied by the j^{th} level cells may be taken to be geometrically decreasing, so $\delta_j = 4^{1-j}A^j$ for some $A < 1$. Thus

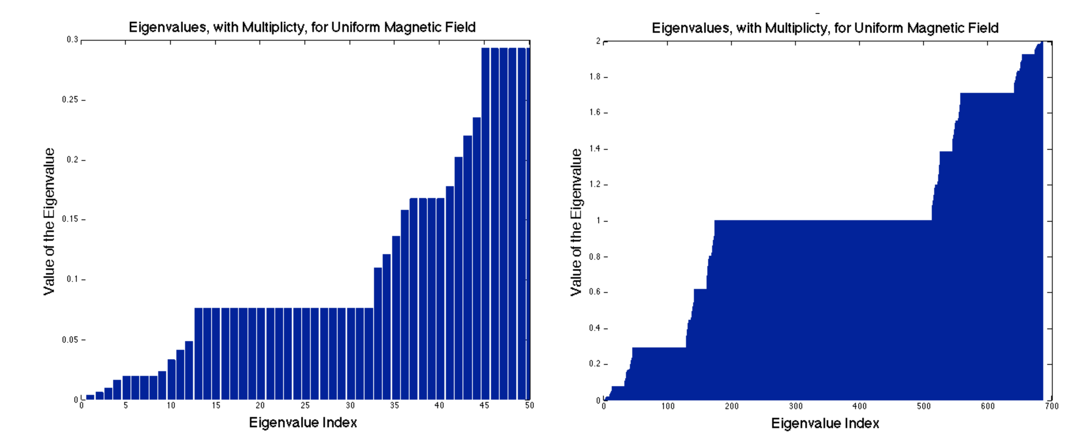
$$\mu_j = 2^{1-2j}\big(A^{n+1} + A^{j+1} - A^j\big).$$

Spectral Plots

Eigenvalue Counting Function of M_n for $n \in \{1,2,3,4\}$



First 50 eigenvalues (left) and full spectrum (right) of M_5



Animations of these plots allow us to see how permissible energy levels of a particle in the diamond fractal vary with the strength of an applied magnetic field.

Acknowledgements

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References

[1] L. Malozemov and A. Teplyaev, Self-similarity, Operators and Dynamics. Mathematical Physics, Analysis and Geometry 6 (2003), 201–218.

[2] J. Bellissard, Renormalization Group Analysis and Quasicrystals. Ideas and Methods in Analysis, Stochastics, and Applications (1992), 119–129.