Fractal Alternating Current Circuits

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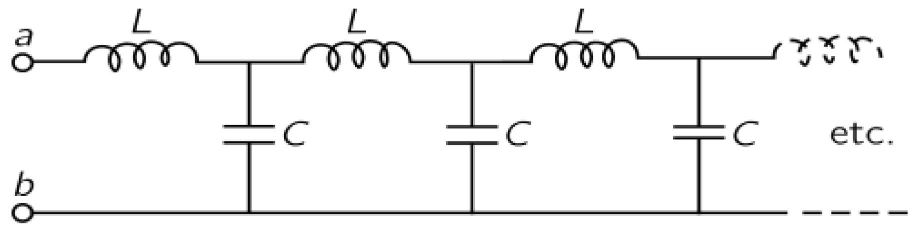
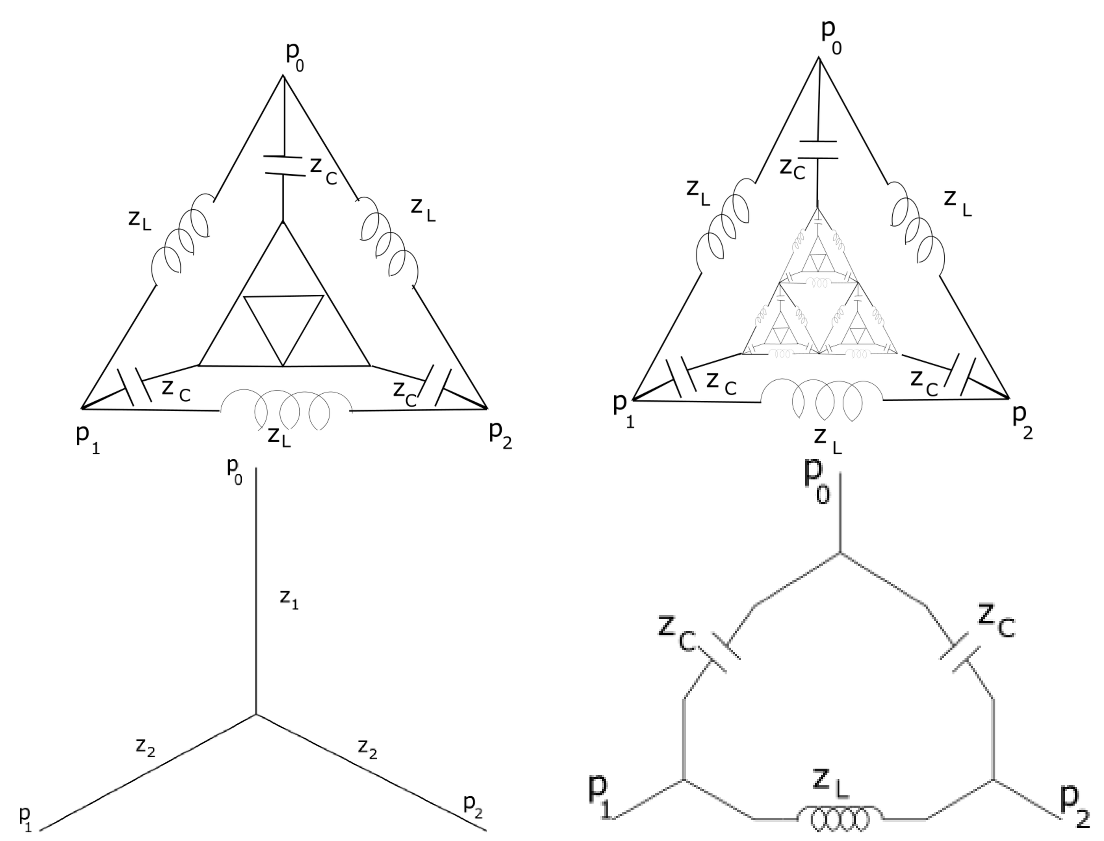
Motivation

Feynman introduces an infinite ladder LC-circuit in his lecture notes [1]. He finds the impedance of the infinite case by assuming the existence of the limit of the impedance as rungs are added to the ladder.



Under certain conditions, this impedance has a positive real part, despite the fact that the circuit is built from components with purely imaginary impedances.

We consider whether this physically interesting situation has analogues for fractal ladder circuits, specifically the self-similar Sierpinski Gasket ladder circuit (top) and the weakly self-similar Hanoi Tower-like circuit (bottom).



These circuits may serve as a foundation for analyzing wave propagation on fractals, e.g. by considering the analogue of the Telegraphers' equations in this setting [2].

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Characteristic Impedance and Filter Conditions

The characteristic impedance z of the SG ladder is the impedance between vertices p_0 and p_1. It can be computed by replacing infinite portions of the ladder with the characteristic impedance and reducing using Kirchoff's laws. The circuit is a filter when it has energy dissipation, i.e. when z has positive real part.

Theorem 1 *If $z_L = i\omega L$ and $z_C = 1/(i\omega C)$, where ω is the AC frequency, then the SG ladder is a filter when $9(4-\sqrt{15}) < 2\omega^2 LC < 9(4+\sqrt{15})$, and*

$$z = \frac{1}{10\omega C}\left(2i\omega^2 LC + 9i + \sqrt{144\omega^2 LC - 4(\omega^2 LC)^2 - 81}\right).$$

In general, the impedance of an infinite LC circuit is not the limit of the impedances of finite approximations to the infinite circuit. However we have the following result.

Theorem 2 *Let $z_{N,\epsilon}$ be the characteristic impedance at the N^{th} stage of construction of the SG ladder with $\epsilon > 0$ added to the impedances. Then,*

$$\lim_{\epsilon\to 0^+}\lim_{N\to\infty} z_{N,\epsilon} = z.$$

The Hanoi circuit has two characteristic impedances, z_1 between p_0 and the central point of the Y-shape and z_2 between p_1 and the central point. An easily stated special case of our results on this circuit is as follows.

Theorem 3 *If we require that impedances are scaled by a factor of $\frac{1}{2}$ from level to level then the Hanoi circuit has characteristic impedances $z_1 = \frac{2}{i\omega C}$ and $z_2 = 2\sqrt{L/C}$.*

Harmonic Functions and Extension Matrices

If potentials $f(p_j)$, $j = 0, 1, 2$, are assigned at the outer vertices of the circuit, then energy minimization gives a unique extension to a function f defined on all vertices. This function is harmonic, meaning that for all x,

$$\Delta f(x) = \sum_{x\sim y}\frac{1}{z_{xy}}\big(f(x) - f(y)\big) = 0.$$

Extending f from one level of the graph to the next is a linear operation and can therefore be represented by a product of harmonic extension matrices.

Theorem 4 *The matrix A, below, takes the potentials at the outer points of a SG ladder circuit to those at the three outer points of the inner triangle element. Using self-similarity, the potentials at all other points can be determined by iteratively applying A and the known $\frac{1}{5}$-$\frac{2}{5}$ rule for the Sierpinski Gasket.*

$$A = \frac{1}{9z_C + 5z}\begin{bmatrix} 3z_C + 5z & 3z_C & 3z_C \\ 3z_C & 3z_C + 5z & 3z_C \\ 3z_C & 3z_C & 3z_C + 5z \end{bmatrix}.$$

The corresponding matrix for the Hanoi graph is unwieldy, but in the special case from Theorem 3 the eigenvalues λ_j and eigenvectors e_j are:

$$\lambda_0 = 1,\ e_0 = \begin{bmatrix}1\\1\\1\end{bmatrix};\ \lambda_1 = \frac{(z_2+z_L)(2z_1+z_2)}{3z_2(2z_1+z_2)+z_L(z_1+2z_2+2z_C)},\ e_1 = \begin{bmatrix}0\\1\\-1\end{bmatrix};\ \lambda_2 = \frac{z_1+z_2+z_C}{3z_1+2z_2},\ \begin{bmatrix}\frac{2z_1}{z_2}\\-1\\-1\end{bmatrix}.$$

Any function on the vertices of the Hanoi is a linear combination of these three harmonic functions, and the intermediate vertices in each level have values that are weighted averages of the vertices already determined.

Power Dissipation

In any AC circuit, complex power is the product of voltage and current. If the potentials at all points in the circuit are stored in a vector Q, the complex power dissipation is defined by

$$P = (EQ)^T(CEQ) = Q^T E^T CEQ$$

where E is a vertex-edge transference matrix sending the potentials at vertices to the potential differences across edges, and C is a diagonal conductance matrix sending edge voltage to edge current according to Ohm's law.

Theorem 5 *The operator $D = E^T CE$ that sends potential to total power dissipation on each circuit is invariant under network reduction upon taking the Schur complement.*

For the SG circuit with characteristic impedance z, the simplest power dissipation operator is

$$D = \frac{1}{z}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

For the Hanoi circuit with characteristic impedances z_1 and z_2, the simplest power dissipation operator is

$$D = \frac{1}{2z_1 + z_2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & \frac{z_1+z_2}{z_2} & -\frac{z_1}{z_2} \\ -1 & -\frac{z_1}{z_2} & \frac{z_1+z_2}{z_2} \end{bmatrix}$$

Real power can be similarly obtained from the real part of potential energy with the operator $\mathrm{Re}(D)$, which is also constant from level to level, confirming that energy is conserved through the infinite substitution process.

We can also use our harmonic extension matrices to discover where in the circuit power is dissipated. This computation yields a complex analogue of the Kusuoka measure for the fractal ladder circuits.

References

[1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on Physics*, vol. 2. Basic Books, California Institute of Technology, 1965-2013.

[2] E. Akkermans, J. P. Chen, G. V. Dunne, L. G. Rogers, and A. Teplyaev, *Fractal AC circuits and propagating waves on fractals*. arXiv: 1507.05682.