Fractal Alternating Current Ladder Circuits

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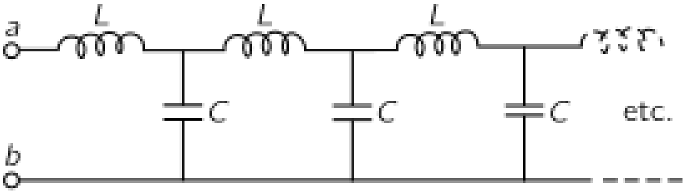
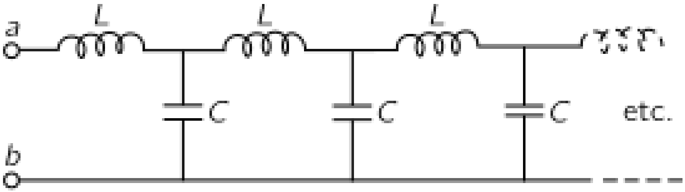
Infinite Ladder Circuit

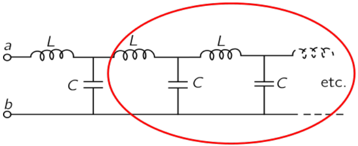


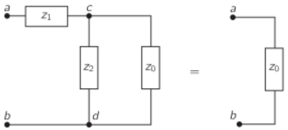
Infinite Ladder Circuit



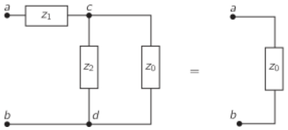
$$z_L = i\omega L$$

$$z_C = \frac{1}{i\omega C}$$

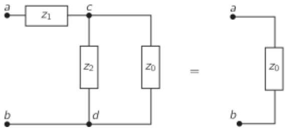

a
L
L
L
C
C
C
etc.
b


a
z1
c
z2
z0
b
d
=
a
z0
b

Characteristic Impedance

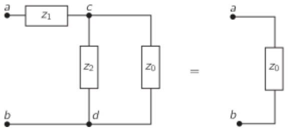


Characteristic Impedance



$$z = \left(\frac{1}{1/(i\omega C)} + \frac{1}{z} \right)^{-1} + (i\omega L)$$

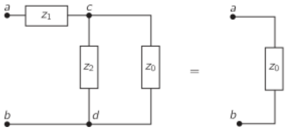
Characteristic Impedance



$$z = \left(\frac{1}{1/(i\omega C)} + \frac{1}{z} \right)^{-1} + (i\omega L)$$

$$z = \frac{1}{2C} \left(i\omega LC + \sqrt{4LC - \omega^2 L^2 C^2} \right)$$

Characteristic Impedance



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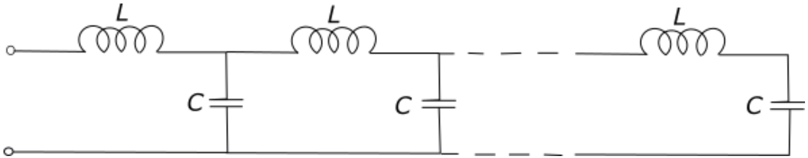
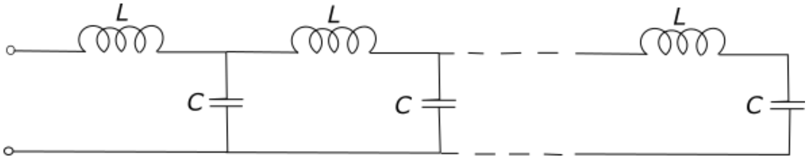
Filter Condition: $\omega^2 LC < 4$.

Finite Approximations



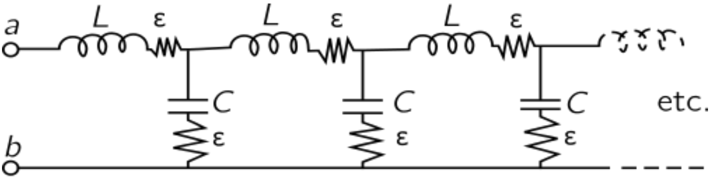
Finite Approximations

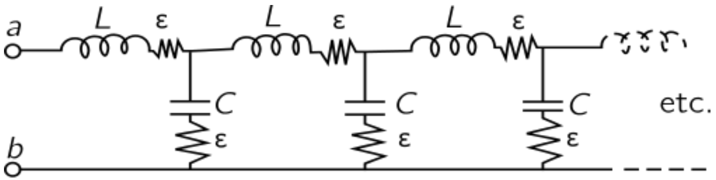


Yoon 2007

Let z_N be the characteristic impedance of the circuit at the N^{th} stage.
Then $\lim_{N \to \infty} z_N$ does not exist.

Finite Approximations



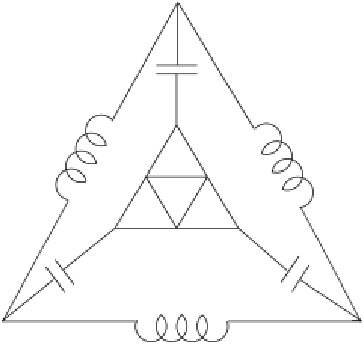


Yoon 2007

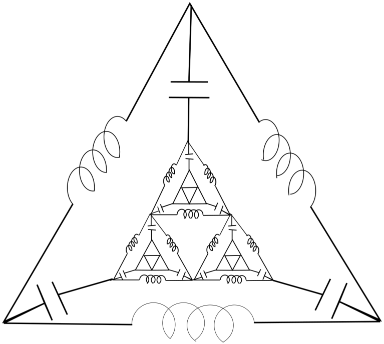
Let $z_{N,\epsilon}$ be the characteristic impedance at the N^{th} stage of construction of the infinite ladder with small real $\epsilon > 0$ added to each impedance. Then,

$$\lim_{\epsilon \to 0^+} \lim_{N \to \infty} z_{N,\epsilon} = z = \frac{1}{2C}\left(i\omega LC + \sqrt{4LC - \omega^2 L^2 C^2}\right)$$

Sierpinski Gasket Circuit Construction

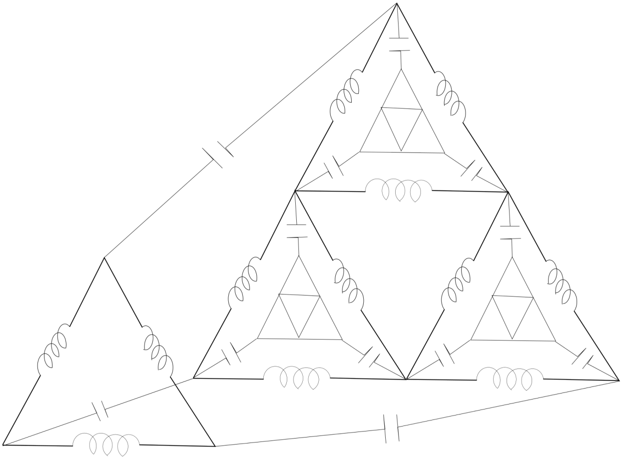


SG circuit level 1



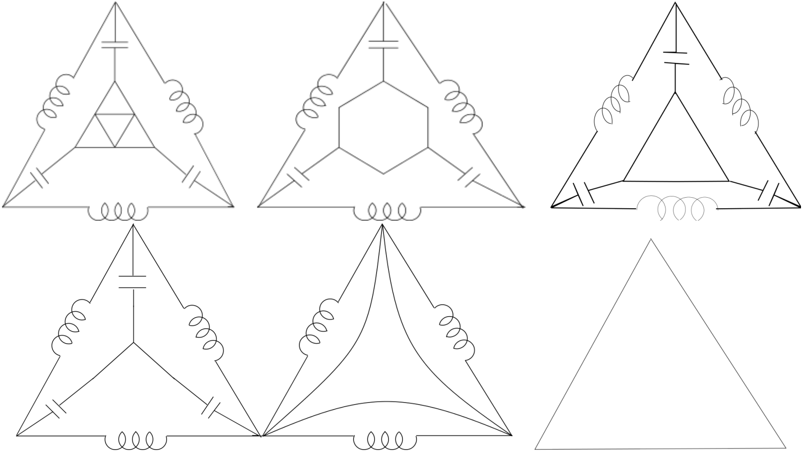
SG circuit level 2

Sierpinski Gasket Circuit Construction



SG circuit level 2

Sierpinski Gasket Circuit Construction



Characteristic Impedance

Characteristic Impedance

Theorem 1

If $z_L = i\omega L$ and $z_C = 1/(i\omega C)$, where ω is the AC frequency, then the SG ladder has characteristic impedance

$$z = \frac{1}{10\omega C}\left(2i\omega^2 LC + 9i + \sqrt{144\omega^2 LC - 4(\omega^2 LC)^2 - 81}\right),$$

and is a filter when

$$9(4 - \sqrt{15}) < 2\omega^2 LC < 9(4 + \sqrt{15}).$$

Finite Approximations

Theorem 2

For finite approximations of the SG ladder,

- i.) $\lim_{N \to \infty} z_N$ does not exist.

Finite Approximations

Theorem 2

For finite approximations of the SG ladder,

- i.) $\lim_{N\to\infty} z_N$ does not exist.
- ii.) $z_{N,\epsilon}$ converges for any $\epsilon > 0$ and

$$\lim_{\epsilon\to 0^+} \lim_{N\to\infty} z_{N,\epsilon} = z,$$

the characteristic impedance of the SG ladder.

Harmonic Functions and Extension Matrices

If potentials $f(p_j)$, $j = 0, 1, 2$, are assigned at the outer vertices of the circuit, then energy minimization gives a unique extension to a function f defined on all vertices. This function is harmonic, meaning that for all x,

$$\Delta f(x) = \sum_{x \sim y} \frac{1}{z_{xy}} \big(f(x) - f(y)\big) = 0.$$

If applied to all points on one level of the graph, this sets up a system of linear equations that allows f to be represented by a product of harmonic extension matrices.

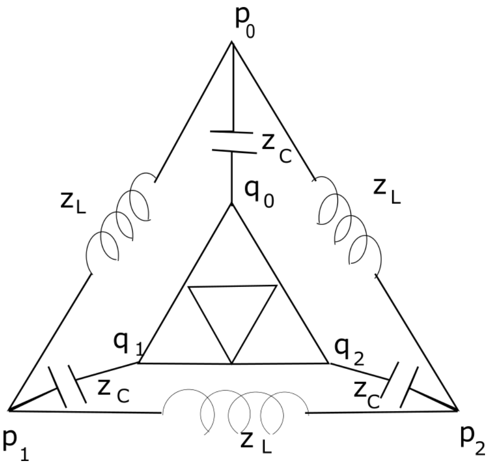
Harmonic Functions and Extension Matrices

Theorem

The matrix A, below, takes the potentials at the outer points of a SG ladder circuit to those at three outer points of the inner triangle element. Using self-similarity, the potentials at all other points can be determined by iteratively applying A and the known $\frac{1}{5}$-$\frac{2}{5}$ rule for the Sierpinski Gasket.

$$A = \frac{1}{9z_C + 5z} \begin{bmatrix} 3z_C + 5z & 3z_C & 3z_C \\ 3z_C & 3z_C + 5z & 3z_C \\ 3z_C & 3z_C & 3z_C + 5z \end{bmatrix}.$$

Harmonic Functions and Extension Matrices



Definition

The $\frac{1}{5}$-$\frac{2}{5}$ rule states that the matrices A_0, A_1, and A_2 send the potentials at the outer corners of the level 2 Sierpinski gasket, $f(q_j)$, $j = 0, 1, 2$, to the potentials at the corners of the top, left, and right cells respectively.

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix} \quad A_1 = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix} \quad A_2 = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Complex Power Dissipation

$$P = IV$$

If the potentials at all points in the circuit are stored in a vector Q, the complex power dissipation is defined by

$$P = (EQ)^T(CEQ) = Q^T E^T CEQ$$

- E is a vertex-edge transference matrix sending the potentials at vertices to the potential differences across edges.
- C is a diagonal conductance matrix sending edge voltage to edge current according to Ohm's law.

Complex Power Dissipation

Theorem

The operator $D = E^T CE$ that sends potential to total power dissipation on each circuit is invariant under network reduction upon taking the Schur complement.

For the SG circuit with characteristic impedance z, the simplest power dissipation operator is

$$D = \frac{1}{z}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

References

[1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on Physics*, vol. 2. Basic Books, California Institute of Technology, 1965-2013.
[2] E. Akkermans, J. P. Chen, G. V. Dunne, L. G. Rogers, and A. Teplyaev, *Fractal AC circuits and propagating waves on fractals*. arXiv: 1507.05682.
[3] S. H. Yoon, *Ladder-type circuits revisited*, vol. 28. Eur. J. Phys., 2007.